Eidgenössische Technische Hochschule Zürich
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Non-Archimedean hyperbolic planes

Bachelor Thesis

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Abstract

Classical geometry satisfies two crucial properties. Through a given point and to a given line, there is a unique parallel line (parallel axiom) and for two distances, it is always possible to take the smaller and add it up finitely many times to exceed the bigger (Archimdes' axiom). The relaxation of the first property has lead to hyperbolic geometry, where there are many parallel lines to a given line through a given point. The negation of Archimedes' axiom leads to an algebraic consideration of planes over arbitrary fields. This thesis discusses three models of geometries, where both axioms are unsatisfied. Building on classical models of the hyperbolic plane, we extend the notion of distance to non-Archimedean base fields. This naturally gives rise to a metric space that is shown to have tree-like properties.

An article by G. W. Brumfiel contains the construction of the distance function and the metric space of a hyperbolic plane over a non-Archimedean field and it is shown with geometrical tools that it is a tree. We elaborate on these ideas and give more detailed proofs for many of the statements (some of them were left as an exercise for the reader by Brumfiel). We will additionally include a classification of ordered fields with many examples. The purely analytic constructions are also compared to the axiomatic desription of geometry by Hilbert.

Contents

Introduction

In his book *Grundlagen der Geometrie*, Hilbert introduced a system of axioms for geometry, organizing the different geometries, mostly the Euclidean plane and the hyperbolic plane. The requirement and independence of the axioms was studied, see [5] for an overview. Later, more exotic combinations of axioms were tried out. Non-Archimedean hyperbolic planes were investigated with algebraic methods, for example by Morgan and Shalen [8]. Brumfiel [2] then provided a geometric proof of the results.

The purpose of this thesis is to give a more accessible explanation of the results in [2]. Brumfiel often declared statements as common knowledge or left the proof to the reader. We will give detailed proofs of the statements, allowing people without profound knowledge about hyperbolic geometry to access this field. In particular chapters 1 and 3 suit well as an introduction to the study of the hyperbolic plane. Throughout, we will focus on Euclidean fields (fields with square roots), as they simplify the proofs. To enhance the understanding, we will also give many examples of ordered fields, as well as a classification to provide tools to come up with new ones in chapter 2. In chapter 4 we state and prove the theorem that the metric space associated to a non-Archimedean hyperbolic plane is a (Λ-)tree. As we start from a field and construct models directly, we will also keep track of which axioms by Hilbert are satisfied.

Chapter 1

Hyperbolic planes and rigid motions

Given any ordered field F (for a definition and an overview of ordered fields, see chapter 2, in this chapter F may be thought of as $\mathbb{R}$ or $\mathbb{Q}$), there are multiple ways to construct a hyperbolic plane. In this chapter three such models are presented. They are equivalent, but in some instances the use of one might be preferred over another. In this chapter, the models as well as their relationships will be given. Then, a group of actions on the hyperbolic plane is identified. Later, the cross ratio, a multiplicative hyperbolic distance, will be defined for two points on the hyperbolic plane. Showing that the hyperbolic distance is invariant under the action, justifies calling the actions isometries of the hyperbolic plane. The formulae in the various models are adapted from [2].

1.1 Models of the hyperbolic plane

Given any ordered field F, the complex plane $F[i]$ for $i^2 = -1$ can be considered. The *upper half plane model* of the hyperbolic plane

$$HF^2 = \{w \in F[i] : \mathrm{Imag}(w) > 0\} \cong \{(u,v) \in F^2 : v > 0\}$$

is the set of complex numbers that have positive imaginary part. Another model of the hyperbolic plane is the *conformal* or *Poincaré disk model*

$$B = \{z \in F[i] : z\bar{z} < 1\} \cong \{(x,y) \in F^2 : x^2 + y^2 < 1\}.$$

These models are connected by a transformation t as stated in the next proposition.

Proposition 1.1 *The Möbius transformation*

$$\begin{aligned} t : HF^2 &\longrightarrow B \\ w = u + iv &\longmapsto \frac{w-i}{w+i} = \frac{u^2+v^2-1}{u^2+(v+1)^2} - \frac{2u}{u^2+(v+1)^2} i \end{aligned} \tag{1.1}$$

is a bijective function with inverse

$$t^{-1}(z) = t^{-1}(x+iy) = i\,\frac{1+z}{1-z} = \frac{-2y + i(1-(x^2+y^2))}{(1+x)^2+y^2}. \tag{1.2}$$

Proof First, t needs to be a well defined function. As $\mathrm{Imag}(w) > 0$, the formula of t defines a function from HF^2 to F^2. One verifies that $t(u+iv) = x+iy$ with

$$x = \frac{u^2+v^2-1}{u^2+(v+1)^2} \quad \text{and} \quad y = -\frac{2u}{u^2+(v+1)^2}.$$

For $t(z)$ to lie in the ball B it needs to satisfy $x^2+y^2 < 1$

$$\begin{aligned} x^2+y^2 &= \frac{(u^2+v^2-1)^2+4u^2}{(u^2+(v+1)^2)^2} = \frac{(u^2+v^2)^2-2(u^2+v^2)+1+4u^2}{(u^2+v^2+2v+1)^2} \\ &= \frac{(u^2+v^2)^2+2u^2-2v^2+1}{(u^2+v^2)^2+2(u^2+v^2)(2v+1)+(2v+1)^2} < 1 \end{aligned}$$

$$\begin{aligned} \iff &\quad (u^2+v^2)^2+2u^2-2v^2+1 < (u^2+v^2)^2+4u^2v+2u^2+4v^3+6v^2+4v+1 \\ \iff &\quad 0 < 4u^2v+4v^3+8v^2+4v = 4v(u^2+(v+1)^2), \end{aligned}$$

which is true since $v > 0$. Thus t is well defined.

Next, the formula of

$$\begin{aligned} t^{-1} : B &\longrightarrow \quad HF^2 \\ z &\longmapsto i\,\frac{1+z}{1-z} \end{aligned}$$

defines a function to $F[i]$ as $z\bar{z} < 1$. We show that it is a well defined function and sends points to the upper half plane HF^2 by setting $z = x+iy \in B$ for $x^2+y^2 < 1$

$$\begin{aligned} t^{-1}(z) = t^{-1}(x+iy) &= \frac{-2y+i(1-x^2-y^2)}{(1-x)^2+y^2} \\ \Rightarrow \mathrm{Imag}\left(t^{-1}(z)\right) &= \frac{1-(x^2+y^2)}{(1-x)^2+y^2} > \frac{(x^2+y^2)-(x^2+y^2)}{(1-x)^2+y^2} = 0, \end{aligned}$$

which shows that $t^{-1}(z)$ is in the upper half plane HF^2. Finally, we show that t^{-1} is an both-sided inverse of t by calculating

$$\begin{aligned} t(t^{-1}(z)) = t(t^{-1}(x+iy)) = t\left(i\frac{1+x+iy}{1-(x+iy)}\right) &= \frac{i\frac{1+x+iy}{1-(x+iy)} - i}{i\frac{1+x+iy}{1-(x+iy)} + i} \\ &= \frac{1+x+iy-(1-(x+iy))}{1+x+iy+1-(x+iy)} = x+iy = z \end{aligned}$$

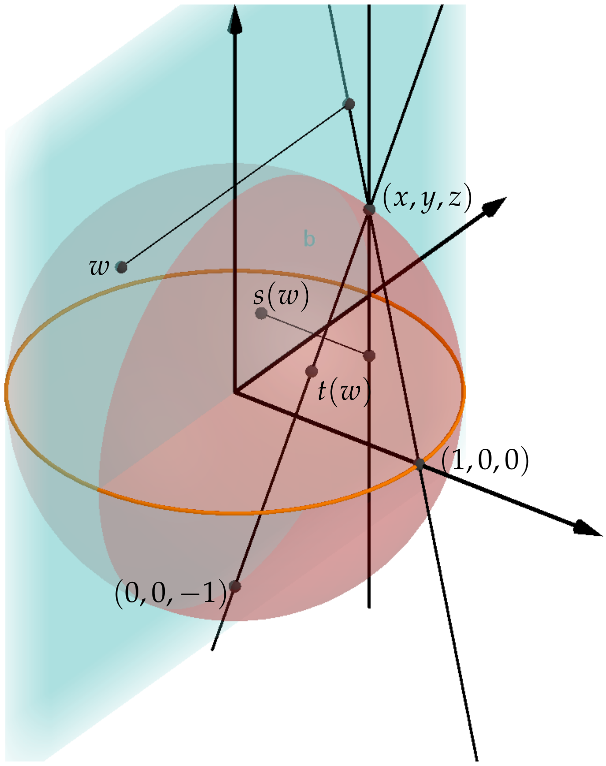

Figure 1.1: The connections between the models of the hyperbolic plane can be visualised in F^3. The point $w \in HF^2$ is mirrored on the third coordinate axis and then stereographically projected to (x, y, z) on the unit sphere. Another stereographic projection then sends it to $t(w) \in B$. Instead projecting (x, y, z) directly downward and mirroring it on the second coordinate axis results in $s(w) = (-x, y) \in B_0$.

This construction is also available online as an interactive GeoGebra visualisation at https://n.ethz.ch/~apraphae/hyperbolic_planes_visualisation.html.

and

$$t^{-1}(t(w)) = t^{-1}(t(u+iv)) = t^{-1}\left(\frac{u+iv-i}{u+iv+i}\right) = i\,\frac{1+\frac{u+iv-i}{u+iv+i}}{1-\frac{u+iv-i}{u+iv+i}}$$
$$= i\,\frac{u+iv+i+u+iv-i}{u+iv+i-(u+iv-i)} = u+iv = w$$

concluding the proof. $\square$

The transformation t extends to the boundaries of HF^2 and B and sends $i, -1, 0, -1 \in HF^2 \cup \partial HF^2$ to $0, i, -1, -i \in B \cup \partial B$ and the real axis in HF^2 to the boundary of B without the point $1 \in B$, which can be thought of as corresponding to 'the point at infinity $i\infty$' in HF^2. The transformation t can be understood as a series of (stereographic) projections as visualized in figure 1.1, where w is sent over a point on the upper half sphere as an intermediate step. This naturally leads to another model of the hyperbolic plane, the *Klein disk model*

$$B_0 = \{(x, y) \in F^2 : \exists z \in F : x^2 + y^2 + z^2 = 1\}.$$

Note that $B_0 \subset B$, but they are not the same in general, since $(x, y) \in B$ is only in B_0 if $\sqrt{1-(x^2+y^2)} \in F$. If it is possible to take square roots in F, then $B = B_0$ as sets, but even then the two models behave differently, as there is a different identification with HF^2 considered in the next proposition.

Proposition 1.2 *The transformation*

$$\begin{aligned} s: \quad HF^2 &\longrightarrow \quad B_0 \\ u + iv &\longmapsto \left(\frac{1-(u^2+v^2)}{1+u^2+v^2}, \frac{-2u}{1+u^2+v^2} \right) \end{aligned} \tag{1.3}$$

is a bijective function with inverse

$$s^{-1}\left((x,y)\right) = \left(\frac{-y}{1+x}, \frac{z}{1+x} \right) \quad \textit{for} \quad z = \sqrt{1-x^2-y^2}. \tag{1.4}$$

Proof To see the that s is well defined, consider an element $u + iv \in HF^2$ and set $s(u+iv) = (x,y)$. For this point to be in B_0 it is necessary to find a z with the property $x^2 + y^2 + z^2 = 1$. Using

$$z = \frac{2v}{1+u^2+v^2} \in F$$

results in

$$x^2 + y^2 + z^2 = \frac{1 - 2(u^2+v^2) + (u^2+v^2)^2 + 4u^2 + 4v^2}{(1+u^2+v^2)^2} = 1,$$

which proves that s is well defined. For s^{-1} note that $z = \sqrt{1-(x^2+y^2)} \in F$ by the definition of B_0 and $z > 0$ by the definition of square root. Now $-1 < x < 1$, so $1 + x > 0$ resulting in

$$\operatorname{Imag}\left(s^{-1}\left((x,y)\right)\right) = \frac{z}{1+x} > 0$$

and $s^{-1}((x,y)) \in HF^2$. To see that s^{-1} is the both-sided inverse of s, one can show with the same $z = \sqrt{1-(x^2+y^2)} \in F$ that

$$s\left(s^{-1}\left((x,y)\right)\right) = (x,y)$$

and

$$s^{-1}\left(s(u+iv)\right) = u + iv$$

concluding the proof that s is a bijection between the two models HF^2 and B_0 of the hyperbolic plane. □

When extended to the boundaries, s sends $i, -1, 0, 1 \in HF^2 \cup \partial HF^2$ to $(0,0), (0,1), (1,0), (0,-1) \in B_0 \cup \partial B_0$, the 'point at infinity $i\infty$' lands at $(-1,0) \in B_0$. Now three models of the hyperbolic plane were presented and naturally the question arises how to get from one of the disk models to the other.

Proposition 1.3 *There is a bijection between the two disk models with formulae*

$$\begin{aligned} ts^{-1} : B_0 &\longrightarrow B \\ (x,y) &\longmapsto \left(\frac{-x}{1+z}, \frac{y}{1+z}\right) \quad \textit{for} \quad z = \sqrt{1-(x^2+y^2)} \end{aligned} \tag{1.5}$$

$$\begin{aligned} st^{-1} : B &\longrightarrow B_0 \\ (x,y) &\longmapsto \left(\frac{-2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}\right) \end{aligned} \tag{1.6}$$

Proof The bijections are obtained by going over the half plane model HF^2 via t and s. Concatenations of bijections are again bijections. One can validate the formulas for ts^{-1} and st^{-1} directly using formulae (1.1), (1.2), (1.4) and (1.3) for t and s. □

Remark 1.4 *These formulae show that the Poincaré and the Klein model of the hyperbolic plane only differ by a dilation by a factor $\frac{1}{1+z}$ centered at the origin (and a reflection on the y-axis). As a result, the limit points on the boundary of B are the same as the ones on the boundary of B_0 and are fixed points of ts^{-1} and st^{-1} (up to the reflection $x \mapsto -x$).*

From now on it makes sense to take a look at a special class of ordered fields, the Euclidean fields.

Definition 1.5 *An ordered field F is called* Euclidean *if every positive element $x > 0 \in F$ is a square: $\exists y \in F : y \cdot y = x \in F$. This element is then called the square root of x ($y = \sqrt{x}$).*

In planes over Euclidean fields the *elementary continuity principle* holds, meaning that all circles and lines in F^2 intersect when they should (when they would intersect in the Cauchy-completion of F), since the equations for lines and circles can be solved in those fields (compare this to the circle-circle intersection property (E) in Appendix A). In general (non-Euclidean) fields this is not the case. For example consider the non-Euclidean field $\mathbb{Q}$. The line that goes through the points $(0,0)$ and $(1,1) \in \mathbb{Q}^2$ does not cut the unit circle. In chapter 2 more properties of ordered fields and their relationships as well as some examples are presented.

Using a Euclidean field F, the three models can be equipped with sets of points called *hyperbolic lines* to create a neutral geometry in the sense of Hilbert (for an overview of Hilbert's axioms, see Appendix A). In the case of HF^2 hyperbolic lines are defined to be half circle arcs or straight vertical rays. (In general planes F^2 a *line* is the set of all points that satisfy a linear equation, a *circle* is the set of all points that satisfy a circle equation.) When extended to the border, the hyperbolic lines would cut the real axis at a right angle. (Right angles are defined on general planes F^2 by saying two vectors $v = (v_1, v_2)$ and $w = (w_1, w_2)$ form a *right angle* if $v \cdot w = v_1w_1 + v_2w_2 = 0$.)

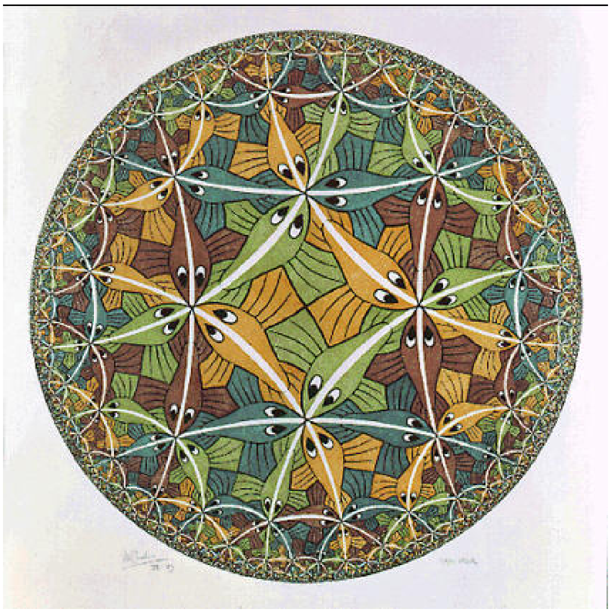

Figure 1.2: An artistic representation of the Poincaré disk B by M.C. Escher. The fish of one color follow a hyperbolic line and all fish have the same size in the hyperbolic metric.

In the Poincaré disk model B, hyperbolic lines are circle segments (as can be seen in figure 1.2) that, when extended to the boundary, cut it at right angles, or line segments that pass through the point $0 \in B$. The fact that the field F is Euclidean is used in the proof that there are intersections with right angles. In the Klein disk model B_0, hyperbolic lines are straight line segments. figure 1.1 illustrates a hyperbolic triangle created by hyperbolic lines in the three models.

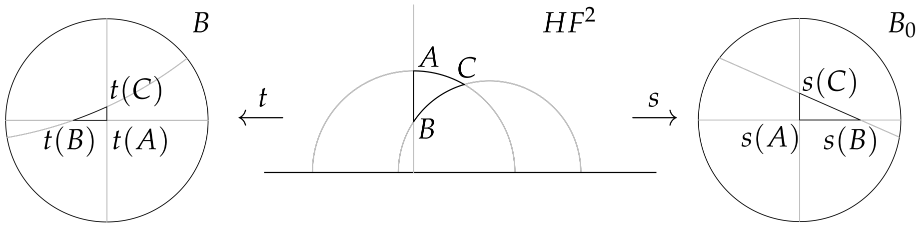

Figure 1.3: The hyperbolic triangle $\triangle ABC$ is limited by three hyperbolic lines. The transformations t and s identify the Poincaré disk B, the half plane HF^2 and the Klein disk B_0 with each other.

It can be shown that these definitions of hyperbolic lines satisfy Hilbert's axioms of incidence (I1 - I3) and betweenness (B1 - B4) (as stated in Appendix A). The hyperbolic lines in the different models remain hyperbolic lines under the identifications t and s from (1.1) and (1.3). For HF^2 and B, this is a result of the fact that t is a Möbius transformation that sends circles or lines to circles or lines. The Klein model B_0 can be understood as a stretching of the Poincaré model B by exactly as much as is needed to get straight lines as stated in remark 1.4. These facts are subjects of lectures about hyperbolic geometry and not proved here.

1.2 Rigid motions

In Hilbert's system of axioms (Appendix A) there is a third group of axioms that are essential to geometry, the congruence axioms (C1 - C6). They require a method to compare sizes of hyperbolic line segments. One way to introduce such a comparison would be to define a metric on the hyperbolic plane. This metric then gives rise to a set of isometries. As the standard definition of the hyperbolic metric for $F = \mathbb{R}$ involves logarithms, the generalization is not straightforward and is subject of chapter 3.1. Instead it is possible to start with a group of isometries that also lead to the axioms of congruence.

Definition 1.6 *The group* $GL_+(2, F) = \{A \in F^{2\times 2} : \det(A) > 0\} \subset GL(2, F)$ *is the group of invertible* 2×2 *matrices over* F *with strictly positive determinant.*

Proposition 1.7 $GL_+(2, F)$ *is a group and if* I *is the unit matrix then its multiples* $F^* = \{\lambda I \in GL_+(2, F) : \lambda \in F\}$ *is the center group of* $GL_+(2, F)$*.*

Proof $GL_+(2, F)$ is a subset of the general linear group. To show that it is a subgroup it is enough to see $\forall A, B \in GL_+(2, F) : AB \in GL_+(2, F)$, which is satisfied since the product of two strictly positive determinants is positive again and $\forall A \in GL_+(2, F) : A^{-1} \in GL_+(2, F)$, which is true since the reciprocal of a positive determinant is positive.

The center group of $GL_+(2, F)$ is the set of all matrices $A \in GL_+(2, F)$ that commute with every other matrix $B \in GL_+(2, F)$. For

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_+(2, F)$$

to be in the center it has to commute with all matrices such as

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B^T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

giving raise to the restrictions

$$\begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix} = AB = BA = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix} \quad \Rightarrow \quad c = 0 \text{ and } a = d$$

$$\begin{pmatrix} a+b & b \\ c+d & d \end{pmatrix} = AB^T = B^T A = \begin{pmatrix} a & b \\ a+c & b+d \end{pmatrix} \quad \Rightarrow \quad b = 0 \text{ and } a = d.$$

The matrices of the form aI already commute with every other matrix, so F^* is the center group. $\square$

Proposition 1.8 *The group $PGL_+(2,F) = GL_+(2,F)/F^*$ acts non-trivially and transitively on HF^2 by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} w = \frac{aw+b}{cw+d}, \quad w \in HF^2. \tag{1.7}$$

Proof The action is well defined on the whole F^2. The action sends elements $w = u + iv \in HF^2$ with $v > 0$ to points with imaginary part

$$\text{Imag}(Aw) = \text{Imag}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} w \right) = \text{Imag}\left(\frac{aw+b}{cw+d} \right) = \frac{\det(A)v}{(cu+d)^2 + c^2v^2} > 0,$$

which are again in HF^2. To be a group action, it also satsifies identity $\lambda I w = w$ and compatibility

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \left[\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} w \right] = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right] w.$$

The elements of $GL_+(2,F)$ that act trivially on HF^2 form exactly the center group F^*, which got condensed into the neutral element in $PGL_+(2,F)$, so $PGL_+(2,F)$ acts non-trivially. The action is also transitive since there is a matrix $AB^{-1} \in GL_+(2,F)$ that sends any point $u + vi \in HF^2$ over i to any other point $u' + v'i \in HF^2$ as follows

$$B = \begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} u' & v' \\ 0 & 0 \end{pmatrix}$$

$$AB^{-1}(u+iv) = \begin{pmatrix} u' & v' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix}^{-1} w = \begin{pmatrix} u' & v' \\ 0 & 1 \end{pmatrix} i = u' + v'i. \qquad \square$$

Via the identifications t (1.1) and s (1.3), $PGL_+(2,F)$ acts on all models of the hyperbolic plane. These actions are the rigid motions on the hyperbolic plane. They adopt the role of orientation preserving isometries similar to the rotations and translations in the usual Euclidean plane. In fact there is also a subgroup of rotations in $PGL_+(2,F)$ that is further analyzed in the various models by [2]. Two hyperbolic line segments (or hyperbolic angles) then are said to be *congruent* if there is an element in $PGL_+(2,F)$ that sends one onto the other. For F Euclidean, it can be proven (see for example [5]) that this notion of congruence satisfies the axioms of congruence (C1 - C6) for line segments (or angles) as stated in Appendix A.

1.3 The cross ratio

In this section, the multiplicative hyperbolic distance (cross ratio) between two points of the hyperbolic plane over a Euclidean field F is introduced.

This is a way of assigning a number in F to measure the size of a hyperbolic line segment or the hyperbolic distance between two points in the hyperbolic plane. This is however not an additivie distance yet.

Definition 1.9 *Two points A and B in the Poincaré model B define a hyperbolic line with two ideal end points P and Q on the boundary ∂B as in figure 1.4. The* cross ratio *of A* and *B is defined to be*

$$D(A,B) = \frac{|AQ||BP|}{|AP||BQ|}, \tag{1.8}$$

where the notation $|z_1z_2| = \|z_1 - z_2\|$ with the F-valued norm on $F[i]$

$$\|u+iv\| = \sqrt{u^2+v^2} \in F.$$

for the Euclidean distance between two points $z_1, z_2 \in F[i]$ is used.

Proposition 1.10 *The cross ratio D on the Poincaré model B is $PGL_+(2,F)$ invariant.*

Proof First we show that cross ratios, such as the one in (1.8), are invariant under general Möbius transformations

$$\begin{aligned} T : F[i] &\longrightarrow F[i] \\ z &\longmapsto \frac{az+b}{cz+d} \quad \text{for} \quad a,b,c,d \in F[i], d \neq 0 \end{aligned}$$

by using the fact that

$$T(z_1) - T(z_2) = \frac{az_1+b}{cz_1+d} - \frac{az_2+b}{cz_2+d} = \frac{(ad-bc)(z_1-z_2)}{(cz_1+d)(cz_2+d)}$$

to get

$$\frac{|T(A)T(Q)| \cdot |T(B)T(P)|}{|T(A)T(P)| \cdot |T(B)T(Q)|} = \frac{|AQ| \cdot |BP|}{|AP| \cdot |BQ|}.$$

Next, we show that the action of $PGL_+(2,F)$ on B is a Möbius transform. The action of $PGL_+(2,F)$ on B is defined via the identifications (1.1) and (1.2) and the action (1.7) on HF^2.

$$\begin{aligned} PGL_+(2,F) \times B \longrightarrow PGL_+(2,F) \times HF^2 \longrightarrow HF^2 \longrightarrow B \\ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) \longmapsto \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, t^{-1}(z)\right) \mapsto t^{-1}(z') \mapsto z' \end{aligned}$$

The resulting formula

$$\begin{aligned} z' &= t\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} t^{-1}(z)\right) = t\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} i\frac{1+z}{1-z}\right) \\ &= \frac{(c-b+i(a+d))z + (b+c+i(a-d))}{(-(b+c)+i(a-d))z + (b-c+i(a+d))} \end{aligned}$$

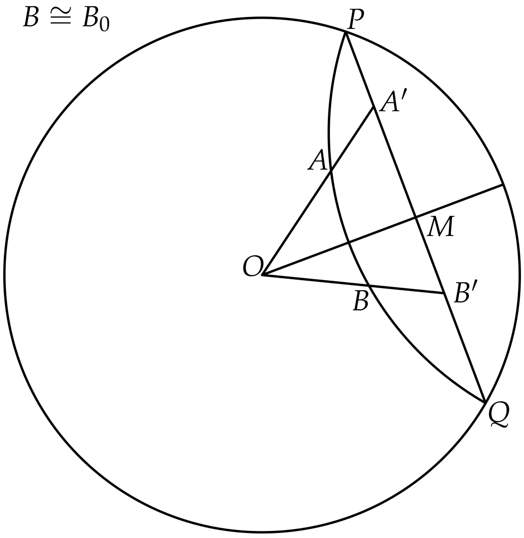

Figure 1.4: The hyperbolic line through A and B in the Poincaré disk model ends in the same points P and Q as the corresponding hyperbolic line through $A' = st^{-1}(A)$ and $B' = st^{-1}(B)$ in the Klein model B_0. The hyperbolic line in the Poincaré model is a Euclidean circle segment, while the hyperbolic line in the Klein model is a Euclidean line segment. The points P, Q, M are also used to verify the formulae for the cross ratio in proposition 1.11. (Note that one of the models has been mirrored as stated in remark 1.4.)

is indeed a Möbius transformation (note that $ad - bc > 0$ implies $b - c + i(a+d) \neq 0$). This implies that the cross ratio (1.8) is $PGL_+(2, F)$-invariant.□

The other models of the hyperbolic plane inherit this cross ratio and the next propositions give some explicit formulae in other models. In the proof of these, the property that $PGL_+(2, F)$ acts transitively on points of the hyperbolic plane (proposition 1.8) can sometimes be used. Namely, the point A can be sent to $0 \in B$ and B can be taken to lie somewhere on the first coordinate axis. In the Klein disk model B_0, the formula looks very similar, only a square root has to be added.

Proposition 1.11 *Let $A', B' \in B_0$ be two points that correspond to the points $A, B \in B$ and let $P, Q \in \partial B$ be the endpoints of the hyperbolic line as shown in figure 1.4. Then the cross ratio is*

$$D(A', B') = \sqrt{\frac{|A'Q||B'P|}{|A'P||B'Q|}} \tag{1.9}$$

in the Klein model B_0 of the hyperbolic plane.

Proof The cross ratio consists of two ratios, and their correspondence

$$\frac{|AQ|}{|AP|} = \sqrt{\frac{|A'Q|}{|A'P|}} \quad \text{and} \quad \frac{|BP|}{|BQ|} = \sqrt{\frac{|B'P|}{|B'Q|}}$$

can be shown individually using formulas 1.6 and 1.5 for st^{-1} and ts^{-1} and trigonometric considerations. To calculate the first ratio introduce the point

M, the middle point of P and Q as in figure 1.4 and the Euclidean distances

$$s = \|P - M\| = \|Q - M\|, \quad c = \|O - M\|, \quad t = \|A' - M\|$$

with the properties

$$s^2 + c^2 = 1, \quad \|Q - A'\| = s + t, \quad \|A' - P\| = s - t$$

hence their naming s for sine and c for cosine (the distances s and t used in this proof should not be confused with the identifications between the models). Using the formula (1.5)

$$A = ts^{-1}(A') = A'\frac{1}{1+z} \quad \text{for} \quad z = \sqrt{1 - \|A'\|} = \sqrt{1 - c^2 - t^2}$$

results in a distance of $t\frac{1}{1+z}$ from the OM-axis and a distance $c\frac{1}{1+z}$ from O aligned with the OM-axis for A, wich leads to the distances

$$|AQ| = \sqrt{\left(c - c\frac{1}{1+z}\right)^2 + \left(s + t\frac{1}{1+z}\right)^2} \quad \text{and}$$
$$|AP| = \sqrt{\left(c - c\frac{1}{1+z}\right)^2 + \left(s - t\frac{1}{1+z}\right)^2}$$

with Pythagoras' theorem. The calculation

$$\begin{aligned}
\frac{|AQ|}{|AP|} &= \frac{\sqrt{c^2\left(1 - \frac{1}{1+z}\right)^2 + \left(s + t\frac{1}{1+z}\right)^2}}{\sqrt{c^2\left(1 - \frac{1}{1+z}\right)^2 + \left(s - t\frac{1}{1+z}\right)^2}} \\
&= \sqrt{\frac{c^2z^2 + (s(1+z)+t)^2}{c^2z^2 + (s(1+z)-t)^2}} \\
&= \sqrt{\frac{1 - c^2 - t^2 + 2s^2z + s^2 + 2st + 2stz + t^2}{1 - c^2 - t^2 + 2s^2z + s^2 - 2st - 2stz + t^2}} \\
&= \sqrt{\frac{s^2 + c^2 - c^2 + 2s^2z + s^2 + 2st + 2stz}{s^2 + c^2 - c^2 + 2s^2z + s^2 - 2st - 2stz}} \\
&= \sqrt{\frac{2s(s + sz + t + tz)}{2s(s + sz - t - tz)}} = \sqrt{\frac{s+t}{s-t}} = \sqrt{\frac{|A'Q|}{|A'P|}}
\end{aligned}$$

for A can also be done for B proving that

$$D(A,B) = \frac{|AQ||BP|}{|AP||BQ|} = \sqrt{\frac{|A'Q||B'P|}{|A'P||B'Q|}}$$

is indeed a formula for the cross ratio in the Klein model B_0. □

There is a formula for the cross ratio in the upper half plane model HF^2. To prove it, we make use of the $PGL_+(2,F)$ invariance.

Proposition 1.12 *Let A, B be two points in the Poincaré disk model with their corresponding points $z_1 = t^{-1}(A), z_2 = t^{-1}(B)$ in the half plane model HF^2. The formula*

$$D(A,B) = \frac{1+t}{1-t} \quad \textit{for} \quad t(z_1,z_2) = \left\| \frac{z_1 - z_2}{\overline{z}_1 - z_2} \right\| \tag{1.10}$$

for the cross ratio holds.

Proof First we show that this formula 1.10 is also invariant under $PGL_+(2,F)$ by showing a stronger result, namely that it is $GL(2,F)$-invariant. Use the fact that $GL(2,F)$ is generated by matrices of the form

$$\begin{aligned}
u(x) &= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} && \text{for} \quad x \in F \\
\delta(\lambda_1,\lambda_2) &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} && \text{for} \quad \lambda_1,\lambda_2 \in F \setminus \{0\} \\
\omega &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\end{aligned}$$

as can be seen by the decompositions

$$\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} = \omega \cdot \delta\left(c,b\right) \cdot u\left(\frac{b}{c}\right), \quad \text{if} \quad a = 0, \quad \left(\text{as } \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0 \Rightarrow c \neq 0\right)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \omega \cdot u\left(\frac{c}{a}\right) \cdot \omega \cdot \delta\left(a, d - abc\right) \cdot u\left(\frac{b}{a}\right), \quad \text{if} \quad a \neq 0.$$

Since the number $t(z_1,z_2) \in F$ is invariant under those matrices

$$\begin{aligned}
t(u(x)z_1, u(x)z_2) &= \left\| \frac{z_1 + x - (z_2 + x)}{\overline{z_1 + x} - (z_2 + x)} \right\| = t(z_1,z_2) \\
t\left(\delta(\lambda_1,\lambda_2)z_1, \delta(\lambda_1,\lambda_2)z_1\right) &= \left\| \frac{\frac{\lambda_1}{\lambda_2}z_1 - \frac{\lambda_1}{\lambda_2}z_2}{\frac{\lambda_1}{\lambda_2}\overline{z}_1 - \frac{\lambda_1}{\lambda_2}z_2} \right\| = t(z_1,z_2) \\
t(\omega z_1, \omega z_2) &= \left\| \frac{\frac{1}{z_1} - \frac{1}{z_2}}{\frac{1}{\overline{z}_1} - \frac{1}{z_2}} \right\| = t(z_1,z_2),
\end{aligned}$$

it is invariant under all matrices in $GL_+(2,F) \subset GL(2,F)$. Through the projection $GL_+(2,F) \twoheadrightarrow PGL_+(2,F) = GL_+(2,F)/F^*$ this property is kept.

Together with the invariance of the cross ratio (proposition 1.10) this implies that we can assume without loss of generality that $A = 0$ and $B = x$ on the positive part of the first coordinate axis. We use equation 1.2 for t^{-1} to get

$$z_1 = t^{-1}(A) = i \qquad z_2 = t^{-1}(B) = i\frac{1+x}{1-x}$$

and

$$t(z_1, z_2) = \left\| \frac{z_1 - z_2}{\overline{z}_1 - z_2} \right\| = x.$$

This proves the proposition with

$$D(A, B) = \frac{|AQ||BP|}{|AP||BQ|} = \frac{1+x}{1-x} = \frac{1+t}{1-t}. \qquad \square$$

The formula of the cross ratio on HF^2 implies one more form on B itself.

Proposition 1.13 *Given two points $z_1, z_2 \in HF^2$ with cross ratio $D(z_1, z_2)$ that are sent to $z_1' = t(z_1)$, $z_2' = t(z_2) \in B$, the cross ratio can be calculated in B as*

$$D(z_1', z_2') = \frac{1 + t'(z_1', z_2')}{1 - t'(z_1', z_2')} \quad \textit{for} \quad t'(z_1', z_2') = \left\| \frac{z_1' - z_2'}{1 - \overline{z_1'} z_2'} \right\|. \tag{1.11}$$

Proof The points are related via t^{-1} from (1.2) as

$$z_1 = i\frac{1 + z_1'}{1 - z_1'}, \quad z_2 = i\frac{1 + z_2'}{1 - z_2'} \in HF^2$$

and inserting z_1 and z_2 into the t from the HF^2-formulation of the cross ratio (1.10) in HF^2 gives the same value

$$t(z_1, z_2) = \left\| \frac{i\frac{1+z_1'}{1-z_1'} - i\frac{1+z_2'}{1-z_2'}}{-i\frac{1+\overline{z_1'}}{1-\overline{z_1'}} - i\frac{1+z_2'}{1-z_2'}} \right\| = \left\| \frac{z_1' - z_2'}{1 - \overline{z_1'} z_2'} \right\| = t'(z_1', z_2')$$

as the t' of the formula of the cross ratio in B. This proves that we found another formula for the cross ratio on B. $\square$

The following properties of D, together with the $PGL_+(2, F)$-invariance, justify the name *multiplicative distance* for the cross ratio.

Proposition 1.14 *Given three points A, B, C in the hyperbolic plane, the cross ratio satisfies the following two properties:*

$$D(A, B) \geqslant 1, \quad \textit{with} \quad D(A, B) = 1 \iff A = B \tag{1.12}$$

$$D(A, B)D(B, C) \geqslant D(A, C), \quad \begin{array}{l}\textit{with equality if and only if } A, B, C \\ \textit{lie on a hyperbolic line in that order.}\end{array} \tag{1.13}$$

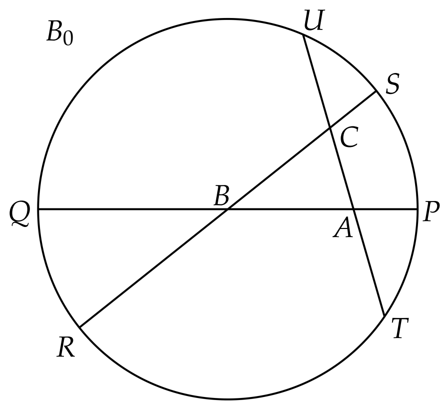

Figure 1.5: Three points A, B, C with their ends P, Q, R, S, U, T on the boundary of B_0 to illustrate the proof of (1.13) $D(A,B)D(B,C) \geqslant D(A,C)$.

Proof Use the formula 1.9 of the Klein model B_0

$$D(A,B) = \sqrt{\frac{|AQ||BP|}{|AP||BQ|}}$$

with

$$|BP| \geqslant |AP| \quad \text{and} \quad |AQ| \geqslant |BQ|$$

to see that

$$D(A,B) \geqslant \sqrt{1} = 1.$$

$$\begin{aligned} &\text{If} \quad A = B \Rightarrow |BP| = |AP| \text{ and } |AQ| = |BQ| \Rightarrow D(A,B) = 1. \\ &\text{If} \quad A \neq B \Rightarrow |BP| > |AP| \text{ and } |AQ| > |BQ| \Rightarrow D(A,B) > 1. \end{aligned}$$

For the proof of (1.13), without loss of generality let $B = 0$ and A on the positive side of the first coordinate axis. Let $QBAP$, $RBCS$ and $UCAT$ be three hyperbolic lines as in figure 1.5. Note that the properties

$$\begin{aligned} |BQ| = |BR| = |BS| &= |BP| = 1 \\ |AT| \geqslant |AP|, &\quad |CU| \geqslant |CS| \\ |AQ| \geqslant |AU|, &\quad |CR| \geqslant |CT| \end{aligned} \tag{1.14}$$

hold. Then

$$(D(A,B)D(B,C))^2 = \frac{|BP||AQ|}{|AP||BQ|} \cdot \frac{|CR||BS|}{|BR||CS|} = \frac{|AQ||CR|}{|AP||CS|} \geqslant \frac{|CT||AU|}{|AT||CU|} = D(A,C)^2$$

shows that $D(A,B)D(B,C) \geqslant D(A,C)$.

If A, B, C lie on a line in that order, then $P = R = T$ and $Q = S = U$ and

$$D(A, B)D(B, C) = \sqrt{\frac{|BP||AQ|}{|AP||BQ|}\frac{|CP||BQ|}{|BP||CQ|}} = \sqrt{\frac{|AQ||CP|}{|AP||CQ|}} = D(A, C).$$

If A, B, C are not collinear, then all the inequalities in (1.14) are strict inequalities. Therefore $D(A, B)D(B, C) > D(A, C)$. $\square$

Chapter 2

Fields

2.1 Ordered fields

The construction of models of the hyperbolic plane in chapter 1 was done for general ordered fields. For the fulfilling of the various axioms of a neutral geometry, F was restricted to be a Euclidean field. This chapter will give an overview over many more properties that fields can have.

Definition 2.1 *[5] A field F with a subset $P \subset F$, whose elements are called* positive elements*, is called an* ordered field *if it satisfies the following two conditions:*

(a) If $a, b \in P$ then $a + b \in P$ and $ab \in P$.

(b) For any $a \in F$, exactly one of the following holds: $a \in P,\ a = 0,\ -a \in P$.

First, to be able to do geometry based on Hilberts axioms (Appendix A), it is advisable to work with ordered fields. In fact, proposition 15.3 in [5] states that if the plane F^2 satisfies the axioms of betweenness (B1 - B4), then F must be an ordered field. The idea is that ordered fields provide a natural order for points on lines such as the coordinate axis $F \cong F \times \{0\} \subset F^2$ satisfiying Hilberts axioms of betweenness. This is is already a considerable limitation on fields F.

Proposition 2.2 *An ordered field is infinite.*

Proof If -1 were in P, then $(-1)(-1) = 1 \in P$, which contradicts condition (a) in the definition of odered fields 2.1. As $0 \neq 1$ it follows that $1 \in P$. The sum of two positive elements such as $1+1, 1+1+1, \ldots$ is positive again. They can be written as elements in $\mathbb{N}$ by setting $n = 1 + \ldots + 1$ (n times). To see that they are all distinct, assume by contradiction that $n = m \in \mathbb{N}$ (and without loss of generality, there are less summands in n than there are in m). Then $0 = m - n = 1 + \ldots + 1$ ($m - n$ times) is supposed to be a positive number, which is a contradiction. So this list contains an infinite amount of elements in F. □

Corollary 2.3 *Every ordered field contains a subring that is isomorphic to* $\mathbb{N}$ *and a subfield that is isomorphic to* $\mathbb{Q}$.

Proof In the previous proof a list of numbers $(1, 1+1, 1+1+1, ...)$ was constructed that is isomorphic to $\mathbb{N}$. Since F is a field, the elements of $\mathbb{Q}$ can be constructed from the ones from $\mathbb{N}$. □

The condition for F to be an ordered field thereby removes the possibility of F to be a finite field. Also some other fields such as $\mathbb{C}$ and (even non-Archimedean ones such as) the p-adic numbers are excluded by this property (for example we have $|p|_p = p$ but $|p+1|_p = 1$ and $|2p+1|_p = 1 < |p|_p + |p+1|_p = 3$, which contradicts the triangle inequality that follows for absolute values from the definition of ordered fields).

2.2 Real closed fields, Archimedean fields, examples

To make sure that two hyperbolic lines meet where they should, the field F was further restricted to be Euclidean (square roots exist) in chapter 1. However there is another property that leads to similar results.

Definition 2.4 *An ordered field F is said to be* real closed *if every positive element of F has a square root in F and every polynomial of odd degree with coefficients in F has at least one root in F.*

With this property, real closed fields are even more like the real numbers $\mathbb{R}$. An example for an ordered field that is neither Euclidean nor real closed is $\mathbb{Q}$. The field of *constructible numbers* (numbers that can be represented by a finite amount of additions, subtractions, multiplications, divisions and square roots) corresponds to the numbers that can be constructed with a straightedge and a compass. The field of constructible numbers is Euclidean but not real closed as there is no solution to $X^3 = 2$. Examples for real closed fields are the *real algebraic numbers* (the real solutions of rational polynomials) or just $\mathbb{R}$.

Definition 2.5 *An ordered field F is said to be an* Archimedean *field if for all elements* $a \in F$ *there is a natural number (using the embedding of* $\mathbb{N}$ *from corollary 2.3)* $n \in F$ *with* $n > a$.
Otherwise it is called non-Archimedean.

All the previous examples are Archimedean fields. All of them are subfields of $\mathbb{R}$, in fact:

Theorem 2.6 *(Hölder, 1901)*
Every ordered Archimedean field is isomorphic to a subfield of $\mathbb{R}$.

Proof A proof can be found in [9]. □

Next, some examples of non-Archimedean fields shall be given. First, let $\mathbb{Q}(X)$ be the field of *rational functions* of the form $p(x)/q(x)$ for two polyinomials p and $q \neq 0$ with coefficients in $\mathbb{Q}$. The rational numbers $\mathbb{Q}$ are viewed as the subfield of constant polynomials. There are multiple possible orders on this field of functions. For instance, it is possible to look at small positive regions (germs) of functions. An element then is said to be positive if there is $\varepsilon > 0$ such that the function is strictly positive on the open interval $(0, \varepsilon) \subset \mathbb{Q}$. This is possible, since elements in $\mathbb{Q}(X)$ are piecewise continuous (with only finitely many poles and zeroes). Note that this satisfies definition 2.1 and thus is an ordered field. With this definition, the function $X^{-1} \in \mathbb{Q}(X)$ is greater than any constant function (such as the natural numbers) since it tends to infinity when approaching 0 from above. This makes the field $\mathbb{Q}(X)$ a non-Archimedean field. It is not Euclidean (nor real closed) as it faces the same problem as $\mathbb{Q}$ ($\sqrt{2} \notin \mathbb{Q}$). But also the field of real rational functions $\mathbb{R}(X)$ is not Euclidean since $X \in \mathbb{R}(X)$ does not have a square root.

To give an example of a non-Archimedean ordered field that is Euclidean (and even real closed), introduce the *Levi-Civita field* that can be constructed as the set of formal series of the form

$$\sum_{i \in \mathbb{Q}} a_i X^i,$$

where $a_i \in \mathbb{R}$, with the restriction that the support $\{i \in \mathbb{Q} : a_i \neq 0\}$ has to be leftfinite, i.e. for every index there is only a finite amount of other indices that are smaller than it. The multiplication

$$\sum_{i \in \mathbb{Q}} a_i X^i \cdot \sum_{j \in \mathbb{Q}} b_j X^j = \sum_{i+j \in \mathbb{Q}} a_i b_j X^{i+j}$$

is then well defined since for every index $e \in \mathbb{Q}$, there are only finitely many pairs (i, j) with $i + j = e$ (otherwise there would be an infinite decreasing sequence of i's or j's). The Levi-Civita field can be proven to be real closed and it is even the smallest non-Archimedean, Cauchy-complete and real closed extension of $\mathbb{R}$ [10]. A similar field is the set of *Hahn-series*, where the condition to be leftfinite is replaced with well-ordered, i.e. every subset of the support has to have a smallest element. This field was first constructed by Hahn to prove a result, similar to Hölders theorem 2.6, a classification theorem for general (possibly non-Archimedean) fields.

Theorem 2.7 *(Hahn embedding theorem, Hahn 1907)*
An ordered field F is isomorphic to a subring of

$$\mathbb{R}^{F/\sim}, \quad \textit{where} \quad f \sim g \quad \Leftrightarrow \quad \exists n, m \in \mathbb{N} : n|f| > |g| \textit{ and } m|g| > |f|$$

with lexicographical ordering.

Proof A proof can be found in [9]. $\square$

For example, the Archimedean equivalence classes in the field of rational functions $\mathbb{R}(X)$ are the ones that start with ..., $X^{-1}, 1, X, X^2, \ldots$ and so $\mathbb{R}(Z)/\sim = \mathbb{Z}$. Thus, the Hahn embedding theorem shows that $\mathbb{R}(X)$ has to be a subring of $\mathbb{R}^{\mathbb{Z}} \cong \{(a_i)_{i\in\mathbb{Z}} : a_i \in \mathbb{R}\}$ with the lexicographic ordering.

Another ordered field is the set of *hyperreal numbers* $\mathbb{R}_\omega$, which form the basis for non-standard analysis. The hyperreal numbers can be constructed as equivalence classes of series of real numbers. The equivalence relation depends on the notion of a *non-principal ultrafilter* $\omega : \mathcal{P}(\mathbb{Z}) \to \{0,1\}$ satisfying

$$\begin{aligned}
\omega(\emptyset) &= 0 \\
\omega(\mathbb{Z}) &= 1 \\
\forall A, B \subset \mathbb{Z} &: A \cap B = \emptyset \Rightarrow \omega(A \cup B) = \omega(A) + \omega(B)
\end{aligned}$$

and to make it non-principal

$$\forall A \subset \mathbb{Z} : |A| < \infty : \omega(A) = 0.$$

Note that ultrafilters are similar to measures with only values in $\{0,1\}$, but only finite additivity is required (instead of σ-additivity). One way to construct ultrafilters, is to pick a number $n \in \mathbb{Z}$ and then we say that $\omega(A) = 1 \iff n \in A \subset \mathbb{Z}$. This is an ultrafilter, but it is not non-principal since $\omega(\{n\}) = 1$. To create a non-principal ultrafilter, one might be tempted to define $\omega(A) = 1 \iff |A| = \infty$, but then there are disjoint sets $A, B \subset \mathbb{Z}$, both infinite, (for example the even and odd numbers) that would lead to $\omega(A \cup B) = 2$. No concrete constructions of non-principal ultrafilters are known, but the existence of ultrafilters with this property can be proven via the ultrafilter lemma that uses Zorn's lemma [4]. For a given non-principal ultrafilter ω, the hyperreal numbers are then defined to be equivalence classes of infinite sequences $\mathbb{R}_\omega = \mathbb{R}^{\mathbb{N}}/\sim$, where the equivalence is given by

$$\begin{aligned}
(x_i)_{i\in\mathbb{Z}} = x \sim y = (y_i)_{i\in\mathbb{Z}} \qquad &\text{if} \quad \omega\{i \in \mathbb{Z} : x_i \neq y_i\} = 0 \\
&\text{or} \quad \omega\{i \in \mathbb{Z} : x_i = y_i\} = 1.
\end{aligned}$$

This gives rise to the 0 element

$$[0] = \left\{ x \in \mathbb{R}^{\mathbb{Z}} : \omega\{i \in \mathbb{Z} : x_i = 0\} = 1 \right\} \in \mathbb{R}_\omega$$

and all the other elements satisfy $\omega\{i \in \mathbb{Z} : x_i = 0\} = 0$. For $0 \neq x = (x_i)_{i\in\mathbb{Z}}$, define y by

$$y_i = \begin{cases} x_i, & \text{if } x_i \neq 0 \\ 1, & \text{if } x_i = 0 \end{cases} \qquad \text{for all } i \in \mathbb{Z}$$

and note that $x \sim y$. This then allows to define the multiplicative inverse (componentwise), turning $\mathbb{R}_\omega$ into a field. There are many more ordered non-Archimedean fields that could be cited here. For example the *surreal numbers*, see [3] and [6], are particularly interesting since they are in some sense the largest ordered fields.

2.3 Microbial fields

To conclude this chapter, one more property of fields is explored. This property will become important for the definition of a logarithm in the next chapter.

Definition 2.8 *A positive element $b > 0$ of an ordered field F is called a* big element *if*

$$\forall f \in F\ \exists n \in \mathbb{N} : b^n > f$$

and the reciprocial of a big element is called a microbe*. An ordered field F is called a* microbial field*, if it contains a big element (or a microbe).*

In $\mathbb{Q}$ or $\mathbb{R}$, every number that is bigger than 1 is a big element. Numbers between 0 and 1 are microbes. Together with Hölder's theorem 2.6 this implies that all Archimedean fields are microbial fields. In the non-Archimedean case, the embedded rational numbers are never big elements. Some non-Archimedean fields, like the field of rational functions or the Levi-Civita field have a big element X^{-1} and thus are microbial fields. Other fields like the hyperreals don't have a big element. Any finite power of the element $x = (x_i)_{i\in\mathbb{Z}} \in \mathbb{R}_\omega$ is still smaller than the element $x' = (x_i^i)_{i\in\mathbb{Z}} \in \mathbb{R}_\omega$. So, the hyperreals do not form a microbial field. Given a microbial field, more microbial fields can be constructed by field extensions of finite transcendence degree.

Theorem 2.9 *Let $F \subset K$ be a field extension with finite transcendence degree, where F is a microbial field and K is an ordered field. Then K is also a microbial field.*

Proof [2]
If K is Archimedean over F ($\forall k \in K\ \exists f \in F : \frac{1}{f} < k < f$), then a big element in F ($b \in F : \forall f \in F\ \exists n \in \mathbb{N} : b^n > f$) is also big in K. If however K is not Archimedean over F, then there is a positive element $b \in K$ that is bigger than any element in F. This $b \in K$ is transcendent over F, so the field $F(b)$ can be constructed with b as the big element to get a microbial field. $F(b) \subset K$ is a new microbial field with lower transcendence degree over K. So the claim follows by induction. $\square$

This proof shows that function spaces with finitely many generating variables over microbial fields, such as a Levi-Civita field analogue with more than one variable, are again microbial fields. As $\mathbb{R}_\omega$ is not a microbial field, it also follows that it has infinite transcendence degree over $\mathbb{R}$.

Chapter 3

The hyperbolic distance

3.1 Logarithms

This section assumes F to be a microbial field, such as $\mathbb{R}$, $\mathbb{R}(X)$ or the Levi-Civita field, with a big element $b \in F$. The goal is to introduce a real valued logarithm with base b as was done in [1] and [2]. Recall the definition of the real numbers $\mathbb{R}$ as the set of all Dedekind cuts of $\mathbb{Q}$.

Definition 3.1 *A* Dedekind cut *of* $\mathbb{Q}$ *is a pair of non-empty sets* $A, B \subset \mathbb{Q}$ *such that*

$$A \cup B = \mathbb{Q} \quad \textit{and}$$

$$\forall p \in A, q \in B : p \leq q.$$

Given a number $a > 0 \in F$, the sets

$$A = \left\{ \frac{m}{n} \in \mathbb{Q} : b^m \leq a^n \right\}, \quad B = \left\{ \frac{m}{n} \in \mathbb{Q} : a^n \leq b^m \right\} \tag{3.1}$$

satisfy the definition for a Dedekind cut (since F is a microbial field, A and B are non-empty). This way, they also define a real number that is called the *real valued logarithm* of a, $\log_b(a)$ and has the property that $\forall p \in A, q \in B : p \leq \log_b(a) \leq q$. It is necessary to work with Dedekind cuts, because rational powers such as $b^{\frac{m}{n}}$ might not exist in the field F. The logarithm has the following basic properties.

Proposition 3.2 *For* $0 < a_1 \leq a_2$, $a \in F$ *and big elements* $b, b_1, b_2 \in F$*, the following characteristic properties of logarithms hold:*

$$\log_b(a_1 a_2) = \log_b(a_1) + \log_b(a_2) \tag{3.2}$$

$$\log_b(a_1) \leq \log_b(a_2) \tag{3.3}$$

$$\log_{b_2}(a) = \log_{b_2}(b_1) \log_{b_1}(a) \tag{3.4}$$

Proof The Dedekind cuts of $\log_b(a_1)$ and $\log_b(a_2)$

$$A_1 = \left\{ \frac{m_1}{n} \in \mathbb{Q} : b^{m_1} \le a_1^n \right\} \quad B_1 = \left\{ \frac{m_1}{n} \in \mathbb{Q} : a_1^n \le b^{m_1} \right\}$$
$$A_2 = \left\{ \frac{m_2}{n} \in \mathbb{Q} : b^{m_2} \le a_2^n \right\} \quad B_2 = \left\{ \frac{m_2}{n} \in \mathbb{Q} : a_2^n \le b^{m_2} \right\}$$

can be added to get

$$A_1 + A_2 = \left\{ \frac{m_1}{n} + \frac{m_2}{n} \in \mathbb{Q} : b^{m_1} \le a_1^n \text{ and } b^{m_2} \le a_2^n \right\},$$
$$B_1 + B_2 = \left\{ \frac{m_1}{n} + \frac{m_2}{n} \in \mathbb{Q} : a_1^n \le b^{m_1} \text{ and } a_2^n \le b^{m_2} \right\}.$$

Elements $p \in A_1 + A_2$ satisfy

$$p = \frac{m_1 + m_2}{n} \quad \text{for} \quad b^{m_1} \le a_1^n \quad \text{and} \quad b^{m_2} \le a_2^n \quad \text{for some} \quad m_1, m_2, n \in \mathbb{Z}$$
$$\Rightarrow \quad b^{m_1} b^{m_2} \le a_1^n a_2^n \quad \Rightarrow \quad b^{m_1+m_2} \le (a_1 a_2)^n$$

and the elements $q \in B_1 + B_2$ similarly satisfy

$$q = \frac{m_1 + m_2}{n} \quad \text{for} \quad (a_1 a_2)^n \le b^{m_1+m_2} \quad \text{for some} \quad m_1, m_2, n \in \mathbb{Z}.$$

Therefore $A_1 + A_2, B_1 + B_2$ are the Dedekind cut of the form (3.1) that defines $\log_b(a_1 a_2)$, proving (3.2).

If $a_1 \le a_2$, then also $a_1^n \le a_2^n$ for $n \in \mathbb{N}$ and

$$A_1 = \left\{ \frac{m}{n} \in \mathbb{Q} : b^m \le a_1^n \right\} \subset \left\{ \frac{m}{n} \in \mathbb{Q} : b^m \le a_2^n \right\} = A_2 \quad \text{for} \quad m \in \mathbb{Z} \text{ and } n \in \mathbb{N}$$

shows that $\log_b(a_1) \le \log_b(a_2)$ has to hold $(\forall a \in A_2 : a \le \log_b(a_2))$ proving (3.3). Note here that $\log_b(a_1) = \log_b(a_2)$ does not imply $a_1 = a_2$.

For $i \in \{1, 2\}$, let

$$A_{b_i a} = \left\{ \frac{m_i}{n} \in \mathbb{Q} : b_i^{m_i} \le a^n \right\}, \quad B_{b_i a} = \left\{ \frac{m_i}{n} \in \mathbb{Q} : a^n \le b_i^{m_i} \right\}$$

and

$$A_{b_2 b_1} = \left\{ \frac{m_2}{m_1} \in \mathbb{Q} : b_2^{m_2} \le b_1{}^{m_1} \right\}, \quad B_{b_2 b_1} = \left\{ \frac{m_2}{m_1} \in \mathbb{Q} : b_1{}^{m_1} \le b_2^{m_2} \right\}$$

denote the Dedekind cuts for $\log_{b_i}(a)$ and $\log_{b_2}(b_1)$. Then for elements $p \in A_{b_2 b_1} \cdot A_{b_1 a}$, $q \in B_{b_2 b_1} \cdot B_{b_1 a}$

$$p = \frac{m_2}{m_1} \frac{m_1}{n} \quad \text{for} \quad b_2^{m_2} \le b_1^{m_1} \le a^n,$$
$$q = \frac{m_2}{m_1} \frac{m_1}{n} \quad \text{for} \quad a^n \le b_1^{m_1} \le b_2^{m_2}$$

shows that equation (3.4) is indeed true, concluding the proof of the proposition. $\square$

In Archimedean fields $F \subset \mathbb{R}$ this is the usual logarithm. For non-Archimedean microbial fields F the logarithm behaves in some surprising ways.

Proposition 3.3 *Let F be a non-Archimedean ordered field with big element b and $r > 0 \in \mathbb{Q} \subset F$ (in the sense of corollary 2.3). Then*

$$\log_b(r) = 0. \tag{3.5}$$

Proof F is non-Archimedean, so $\exists a \in F : \forall n \in \mathbb{N} : a > n$ (definition 2.5). The big element b satisfies (by definition 2.8) $\forall f \in F : \exists n \in \mathbb{N} : b^n > f$, in particular this holds for $f = a$, so $\exists n \in \mathbb{N} : b^n > a$ and a is larger than any natural number. The big element b also does not satisfy the Archimedean property since if there were a natural number $m > b$, then $m^n > b^n > a$, which is impossible. Since r^n is a rational number that can be rounded to a natural number,

$$\frac{1}{b} \leq r^n \leq b$$

holds for all $n \in \mathbb{N}$. Assuming that $r \geq 1$, the Dedekind cut of $\log_b(r)$ is

$$\begin{aligned} A &= \left\{ \frac{m}{n} \in \mathbb{Q} : b^m \leq r^n \right\} = (-\infty, 0], \\ B &= \left\{ \frac{m}{n} \in \mathbb{Q} : r^n \leq b^m \right\} = \bigcup_{n \in \mathbb{N}} [\tfrac{1}{n}, \infty) = (0, \infty), \end{aligned}$$

so $\log_b(r) = 0$. In the other case $0 < r \leq 1$, the Dedekind cut

$$\begin{aligned} A &= \left\{ \frac{m}{n} \in \mathbb{Q} : b^m \leq r^n \right\} = \bigcup_{n \in \mathbb{N}} (-\infty, \tfrac{-1}{n}] = (-\infty, 0), \\ B &= \left\{ \frac{m}{n} \in \mathbb{Q} : r^n \leq b^m \right\} = [0, \infty), \end{aligned}$$

looks similar and results in the same

$$\log_b(r) = 0. \qquad \square$$

Proposition 3.4 *Let a, a' in a non-Archimedean field F with big element $b \in F$. Then*

$$\log_b(|a + a'|) \leq \max \left\{ \log_b(|a|), \log_b(|a'|) \right\}, \tag{3.6}$$

where the convention that $\log_b(0) = -\infty$ is introduced.

Proof The triangle inequality implies $|a + a'| \leq |a| + |a'| \leq 2 \max\{|a|, |a'|\}$. If both sides are strictly positive, take the logarithm with (3.3) from proposition 3.2 to get

$$\log_b(|a + a'|) \leq \log_b \left(2 \max \left\{ |a|, |a'| \right\} \right).$$

We use equations (3.2) and (3.3) again to get

$$\log_b(|a+a'|) \leq \log_b(2) + \max\left\{\log_b\left(|a|\right), \log_b\left(|a'|\right)\right\}.$$

The previous proposition 3.3 implies $\log_b(2) = 0$ concluding the proof. $\square$

Remark 3.5 *The properties (3.2) and (3.6) exactly imply that* $v_b : F \to \mathbb{R}$ *with* $v_b(a) = -\log_b(|a|)$ *is a* valuation *with* value group $\Lambda = v_b(F^+) \subset \mathbb{R}$.

To conclude, some concrete calculations in the Levi-Civita field F are given. Take the big element $b = X^{-1} \in F$ as the base for the logarithm. Then

$$\log_{X^{-1}}(X) = -1 \quad \text{and} \quad \log_{X^{-1}}(X^{-m}) = m,$$

because $\left(X^{-1}\right)^m \leq (X^{-m})^1 \leq \left(X^{-1}\right)^m$. More generally, for $r > 0 \in \mathbb{Q}, a \in F$, propositions 3.2 and 3.3 can be used to show

$$\log_b(ra) = \log_b(r) + \log_b(a) = \log_b(a)$$

and since $\log_b(a) + \log_b(a^{-1}) = \log_b(aa^{-1}) = \log_b(1) = 0$,

$$\log_b(a^{-1}) = -\log_b(a). \tag{3.7}$$

Using $\max\{a, a'\} \leq a + a' \leq 2\max\{a, a'\}$ for positive $a, a' > 0 \in F$, we calculate

$$\begin{aligned} \max\left\{\log_b\left(a\right), \log_b\left(a'\right)\right\} &= \log_b\left(\max\{a, a'\}\right) \leq \log_b(a+a') \\ &\leq \log_b\left(2\max\{a, a'\}\right) = \log_b\left(\max\{a, a'\}\right) \end{aligned}$$

$$\Rightarrow \quad \log_b(a+a') = \max\left\{\log_b\left(a\right), \log_b\left(a'\right)\right\}. \tag{3.8}$$

This allows fast calculations such as

$$\begin{aligned} \log_{X^{-1}}\left(5X^3 + 2 + 4X^{-1}\right) &= \max\left\{\log_{X^{-1}}(X^3), \log_{X^{-1}}(2), \log_{X^{-1}}(X^{-1})\right\} \\ &= \max\{-3, 0, 1\} = 1. \end{aligned}$$

3.2 The hyperbolic metric

Using the cross ratio (multiplicative hyperbolic metric) from chapter 1.3, an additive metric can be constructed. Usually (in the Archimedean setting) the logarithm of the cross ratio is a metric on the hyperbolic plane. The goal of the introduction of a logarithm for general microbial fields in chapter 3.1 was to imitate the construction of the metric in the non-Archimedean case. For this chapter, let F be a Euclidean microbial field that is non-Archimedean. The case where F could be non-Euclidean is studied in [2]. We choose a big element in F and denote the logarithm by $\log = \log_b : F^+ \to \Lambda \subset \mathbb{R}$. As seen in (3.4), the choice of the big element only changes the logarithm by a constant factor, which is unimportant in what follows.

In Euclidean fields F, the cross ratio $D(P,Q)$ of two points $P,Q \in HF^2$ is itself a number in F (as the definition of the cross ratio only contains square roots). By (1.12), $D \geq 1$ holds, in particular the cross ratio is positive. This allows the definitoin of

$$\begin{aligned} d : HF^2 \times HF^2 &\longrightarrow \quad \Lambda \subset \mathbb{R} \\ (P,Q) \quad &\longmapsto \log\left(D\left(P,Q\right)\right). \end{aligned} \tag{3.9}$$

Proposition 3.6 *The logarithm of the distance function D satisfies the definition of a* pseudo-metric *on HF^2. That is, for all points $P,Q,R \in HF^2$*

$$d(P,Q) \geq 0, \tag{3.10}$$

$$d(P,Q) = d(Q,P), \tag{3.11}$$

$$d(P,R) \leq d(P,Q) + d(Q,R) \tag{3.12}$$

hold.

Proof The proof relies on the properties of D from proposition 1.14 and the properties of log from propositon 3.2. Combining (1.12) and (3.3) with the fact that 1 is a rational number in proposition 3.3

$$d(P,Q) = \log(D(P,Q)) \geq \log(1) = 0$$

shows the positiveness (3.10). Note that d cannot be positve definite, since there can be many points $P,Q \in HF^2$ with a rational cross ratio. The symmetry (3.11) follows from the symmetry of the cross ratio that can be seen in any of its definitions. Lastly, combining (1.13) with (3.3) and (3.2)

$$\begin{aligned} d(P,R) = \log\left(D\left(P,R\right)\right) &\leq \log\left(D\left(P,Q\right)D\left(Q,R\right)\right) \\ &= \log\left(D\left(P,Q\right)\right) + \log\left(D\left(Q,R\right)\right) = d\left(P,Q\right) + d\left(Q,R\right) \end{aligned}$$

allows us to proove the triangle inequality (3.12). □

Proposition 3.7 *For $P,Q \in HF^2$,*

$$P \sim Q \quad \Leftrightarrow \quad d(P,Q) = 0.$$

defines an equivalence relation on HF^2.

Proof Let $P,Q,R \in HF^2$. Property (1.12) implies

$$d(P,P) = \log(D(P,P)) = \log(1) = 0.$$

The symmetry of the relation follows from the symmetry of d. If $d(P,Q) = 0$ and $d(Q,R) = 0$, then

$$d(P,R) \leq d(P,Q) + d(Q,R) = 0$$

by the triangle inequality and positiveness of d. $\square$

Using this equivalence relation, we define the set of equivalence classes $TF^2 = HF^2/\sim$. The previous propositions show that $d : TF^2 \times TF^2 \to \Lambda \subset \mathbb{R}$ is a *metric* on TF^2. Since D is $PGL_+(2,F)$-invariant by proposition 1.10, d is invariant too. So $PGL_+(2,F)$ acts on TF^2 as a group of d-isometries. In the next chapter, some properties of this metric space are explored. For now, some concrete formulae for d in the various models of the hyperbolic plane shall be given.

Proposition 3.8 *Let $z = x + iy, z' = x' + iy' \in HF^2$. Then*

$$\begin{aligned} d(z,z') &= \log\left(\frac{(x-x')^2 + y^2 + y'^2}{yy'}\right) \\ &= \max\left\{\log\left(\frac{(x-x')^2}{yy'}\right), \log\left(\frac{y}{y'}\right), \log\left(\frac{y'}{y}\right)\right\} \\ &= \log\left(\frac{\|\bar{z} - z'\|^2}{yy'}\right) \\ &= \max\left\{0, \log\left(\frac{\|z - z'\|^2}{yy'}\right)\right\}. \end{aligned}$$

Proof We use definition 1.9 of the cross ratio in HF^2 and the basic properties 3.2 of the logarithm, to show that

$$d(z,z') = \log\left(\frac{1+t}{1-t}\right) = \log\left(\frac{(1+t)^2}{1-t^2}\right) = 2\log(1+t) + \log\left(\frac{1}{1-t^2}\right)$$

for

$$t = \frac{\|z - z'\|}{\|\bar{z} - z'\|}.$$

Since z and z' lie on the upper halfplane, and $\bar{z}$ is the mirror image of z along the real axis, $\|z - z'\| < \|\bar{z} - z'\|$ therefore

$$\begin{aligned} & 0 \leq t = \frac{\|z - z'\|}{\|\bar{z} - z'\|} \leq 1 \quad \text{and} \quad 1 \leq 1 + t \leq 2, \\ \Rightarrow \quad & 0 = \log(1) \leq \log(1-t) \leq \log(2) = 0, \end{aligned}$$

resulting in

$$d(z,z') = 2\log(1+t) + \log\left(\frac{1}{1-t^2}\right) = \log\left(\frac{1}{1-t^2}\right). \tag{3.13}$$

Now

$$\begin{aligned} t &= \frac{\|z-z'\|}{\|\bar{z}-z'\|} = \sqrt{\frac{(x'-x)^2+(y'-y)^2}{(x'-x)^2+(y'+y)^2}} \\ \Rightarrow\ 1-t^2 &= \frac{4yy'}{(x'-x)^2+y'^2+2yy'+y^2} \\ \Rightarrow\ \frac{1}{1-t^2} &= \frac{(x-x')^2+y^2+y'^2}{4yy'} + \frac{1}{2} \end{aligned}$$

and because $y > 0$ and $y' > 0$, both summands are greater than 0, so equation (3.8) can be used to get

$$\begin{aligned} d(z,z') &= \log\left(\frac{1}{1-t^2}\right) = \log\left(\frac{(x-x')^2+y^2+y'^2}{4yy'} + \frac{1}{2}\right) \\ &= \max\left\{\log\left(\frac{(x-x')^2+y^2+y'^2}{4yy'}\right), \log\left(\frac{1}{2}\right)\right\} \\ &= \max\left\{\log\left(\frac{1}{4}\right) + \log\left(\frac{(x-x')^2+y^2+y'^2}{yy'}\right), 0\right\} \\ &= \max\left\{\log\left(\frac{(x-x')^2}{yy'} + \frac{y}{y'} + \frac{y'}{y}\right), 0\right\}. \end{aligned}$$

The three summands in the logarithm are all positive. If there is a rational number $r \in \mathbb{Q}$ with $r \leq \frac{y}{y'}$ or $r \leq \frac{y'}{y}$ then

$$0 = \log(r) \leq \log\left(\frac{y}{y'}\right) \leq \log\left(\frac{(x-x')^2}{yy'} + \frac{y}{y'} + \frac{y'}{y}\right),$$

shows that the logarithm is positive. If however either $\frac{y}{y'}$ or $\frac{y'}{y}$ is a microbe (positive but smaller than any rational number), then the other one is a big element, and then the logarithm is positive too. Either way the logarithm is greater than 0, so

$$\begin{aligned} d(z,z') &= \max\left\{\log\left(\frac{(x-x')^2}{yy'} + \frac{y}{y'} + \frac{y'}{y}\right), 0\right\} \\ &= \log\left(\frac{(x-x')^2+y^2+y'^2}{yy'}\right), \end{aligned}$$

proving the first formula. From that result, use (3.8) again with three positive summands to get the second formula

$$\begin{aligned}
d(z,z') &= \log\left(\frac{(x-x')^2+y^2+y'^2}{yy'}\right) = \log\left(\frac{(x-x')^2}{yy'}+\frac{y}{y'}+\frac{y'}{y}\right)\\
&= \max\left\{\log\left(\frac{(x-x')^2}{yy'}\right), \log\left(\frac{y}{y'}\right), \log\left(\frac{y'}{y}\right)\right\}.
\end{aligned}$$

Then $\|\bar{z}-z'\|^2 = (x-x')^2+(y'+y)^2 = (x-x')^2+y^2+2yy'+y'^2$ gives the third formula

$$\begin{aligned}
d(z,z') &= \log\left(\frac{(x-x')^2+y^2+y'^2}{yy'}\right)\\
&= \max\left\{\log\left(\frac{(x-x')^2+y^2+y'^2}{yy'}\right), 0\right\}\\
&= \max\left\{\log\left(\frac{(x-x')^2+y^2+y'^2}{yy'}\right), \log(2)\right\}\\
&= \log\left(\frac{(x-x')^2+y^2+y'^2}{yy'}+2\right)\\
&= \log\left(\frac{(x-x')^2+y^2+2yy'+y'^2}{yy'}\right) = \log\left(\frac{\|\bar{z}-z'\|^2}{yy'}\right)
\end{aligned}$$

For the last formula, if

$$\frac{\|z-z'\|^2}{yy'} = \frac{(x-x')^2+(y-y')^2}{yy'} = \frac{(x-x')^2+y^2+y'^2}{yy'} - 2$$

is positive, then equation (3.8) from proposition 3.4 can be used twice

$$\begin{aligned}
d(z,z') &= \log\left(\frac{(x-x')^2+y^2+y'^2}{yy'}\right)\\
&= \log\left(\frac{(x-x')^2+y^2+y'^2}{yy'}-2+2\right)\\
&= \max\left\{\log\left(\frac{(x-x')^2+y^2+y'^2}{yy'}-2\right), \log(2)\right\}\\
&= \max\left\{\log\left(\frac{\|z-z'\|^2}{yy'}\right), 0\right\}.
\end{aligned}$$

If however

$$\frac{(x-x')^2+y^2+y'^2}{yy'} - 2 \leq 0,$$

then

$$\frac{(x-x')^2+y^2+y'^2}{yy'} \leq 2$$

and

$$0 \leq d(z,z') = \log\left(\frac{(x-x')^2+y^2+y'^2}{yy'}\right) \leq \log(2) = 0,$$

which verifies the last formula

$$d(z,z') = \max\left\{\log\left(\frac{\|z-z'\|^2}{yy'}\right), 0\right\}. \qquad \square$$

Proposition 3.9 *Let $z = x+iy, z' = x'+iy' \in B$. Then*

$$d(z,z') = \log\left(\frac{\|1-\bar{z}z'\|^2}{(1-\|z\|^2)(1-\|z'\|^2)}\right)$$

Proof As in the last proof equation (3.13) holds again

$$d(z,z') = \log(D(z,z')) = \log\left(\frac{1+t'}{1-t'}\right) = \log\left(\frac{(1+t')^2}{1-t'^2}\right) = \log\left(\frac{1}{1-t'^2}\right)$$

since $1 \leq 1+t' \leq 2$. For

$$t' = \frac{\|z-z'\|}{\|1-\bar{z}z'\|} = \sqrt{\frac{(x-x')^2+(y-y')^2}{(1-xx'-yy')^2+(x'y-xy')^2}},$$

the formula follows from the calculation

$$\begin{aligned} d(z,z') &= \log\left(\frac{1}{1-t'^2}\right) = \log\left(\frac{\|1-\bar{z}z'\|^2}{\|1-\bar{z}z'\|^2 - \|z-z'\|^2}\right) \\ &= \log\left(\frac{\|1-\bar{z}z'\|^2}{(1-\|z\|^2)(1-\|z'\|^2)}\right). \end{aligned} \qquad \square$$

Proposition 3.10 *Let $P = (x,y), Q = (x',y') \in B_0$ with $z = \sqrt{1-x^2-y^2}$ and $z' = \sqrt{1-x'^2-y'^2}$. Then*

$$d(P,Q) = \log\left(\frac{1-(xx'+yy')}{zz'}\right).$$

Proof The strategy is to use the first formula of proposition 3.8 for points $u+iv, u'+iv' \in HF^2$ that are sent to $P, Q \in B_0$ via the identification (1.4)

$$(x,y,z) = \left(\frac{1-(u^2+v^2)}{1+u^2+v^2}, \frac{-2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}\right).$$

Use

$$\frac{1-(xx'+yy')}{zz'} = \frac{(u-u')^2+v^2+v'^2}{2vv'}$$

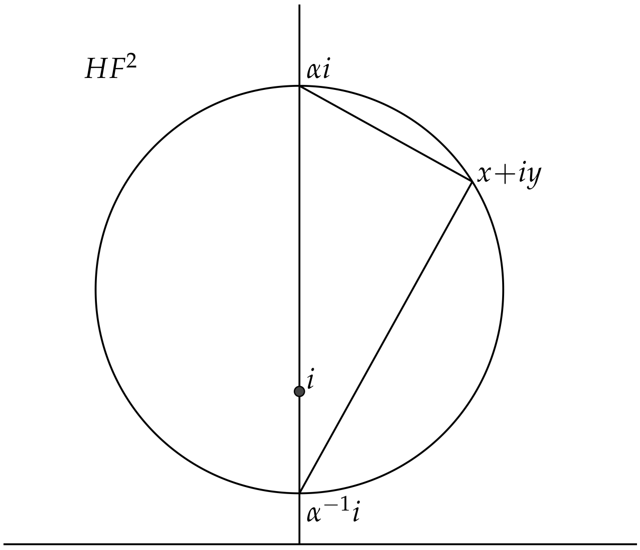

Figure 3.1: Points $x+iy \in HF^2$ with the property that $D(i, x+iy) = \alpha$, lie on a Euclidean circle centered at $\frac{1}{2}(\alpha + \alpha^{-1})$ with radius $\frac{1}{2}(\alpha - \alpha^{-1})$.

in

$$\log\left(\frac{1-(xx'+yy')}{zz'}\right) = \log\left(\frac{(u-u')^2+v^2+v'^2}{vv'}\right) + \log\left(\frac{1}{2}\right) = d(P,Q)$$

to get the formula for d in B_0. □

Lemma 3.11 *The points $z \in HF^2$ that satisfy $D(i,z) = \alpha$ lie on a (Euclidean) circle centered at $\frac{1}{2}(\alpha+\alpha^{-1})i \in HF^2$ with radius $\frac{1}{2}|\alpha - \alpha^{-1}|$ as in figure 3.1.*

Proof Let $z = x + iy \in HF^2$ and

$$\alpha = D(i,z) = \frac{1+t}{1-t} = \frac{1+\frac{\|i-z\|}{\|-i-z\|}}{1-\frac{\|i-z\|}{\|-i-z\|}} = \frac{\sqrt{x^2+(1+y)^2}+\sqrt{x^2+(1-y)^2}}{\sqrt{x^2+(1+y)^2}-\sqrt{x^2+(1-y)^2}}$$

using the defintion of the cross ratio on HF^2 (1.10). In the following let $\alpha > 1$ without loss of generality. Use this to calculate the position of the middle point

$$\frac{\alpha+\frac{1}{\alpha}}{2} = \frac{x^2+y^2+1}{2y}$$

and the radius

$$\frac{\alpha-\frac{1}{\alpha}}{2} = \frac{1}{2y}\sqrt{x^4+y^4+2x^2y^2+2x^2-2y^2+1}$$

for later use. To check whether or not z lies on the circle, we calculate its Euclidean distance from the center

$$\left\| z - \frac{\alpha + \frac{1}{\alpha}}{2} i \right\| = \sqrt{x^2 + \left(y - \frac{\alpha + \frac{1}{\alpha}}{2} \right)^2} = \sqrt{x^2 + \left(y - \frac{x^2 + y^2 + 1}{2y} \right)^2}$$
$$= \frac{1}{2y} \sqrt{x^4 + y^4 + 2x^2y^2 - 2y^2 + 2x^2 + 1} = \frac{\alpha - \frac{1}{\alpha}}{2}$$

and see that it is indeed the claimed quantity. □

Proposition 3.12 *The hyperbolic distance of a point*

$$Mi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} i = \frac{ai + b}{ci + d} \quad \textit{for} \quad M \in PGL_+(2, F)$$

to the point $i \in HF^2$ is

$$d(i, Mi) = \max\left\{ \log(a^2), \log(b^2), \log(b^2), \log(d^2) \right\} - \log(\det(M)).$$

Proof [2]
Calculate the coordinates of the point

$$Mi = \frac{ai + b}{ci + d} = \frac{ac + bd}{c^2 + d^2} + \frac{\det(M)}{c^2 + d^2} i = x + iy.$$

Let $D(i, Mi) = \alpha \geq 1$. By the previous lemma 3.11, the point Mi lies on the circle that goes through αi and $\alpha^{-1} i$ and is centered on the imaginary axis. A non-Archimedean version of Thales' theorem can now be used to show that there is a right angle between the lines $\overline{\alpha i\, Mi}$ and $\overline{\alpha^{-1} i\, Mi}$ as drawn in figure 3.1. The scalar product

$$\begin{pmatrix} x \\ y - \alpha \end{pmatrix} \cdot \begin{pmatrix} x \\ y - \alpha^{-1} \end{pmatrix} = x^2 + (y - \alpha)(y - \alpha^{-1}) = x^2 + y^2 - \alpha y - \alpha^{-1} y + \frac{\alpha}{\alpha}$$
$$= \left(\frac{ac + bd}{c^2 + d^2} \right)^2 + \left(\frac{\det(M)}{c^2 + d^2} \right)^2 + 1 - \left(\alpha + \frac{1}{\alpha} \right) \frac{\det(M)}{c^2 + d^2}$$
$$= \frac{a^2 + b^2 + c^2 + d^2 - \left(\alpha + \frac{1}{\alpha} \right) \det(M)}{c^2 + d^2}$$

has to be zero, resulting in

$$\frac{a^2 + b^2 + c^2 + d^2 - \left(\alpha + \frac{1}{\alpha} \right) \det(M)}{c^2 + d^2} = 0$$
$$\Rightarrow \quad a^2 + b^2 + c^2 + d^2 - \left(\alpha + \frac{1}{\alpha} \right) \det(M) = 0$$
$$\Rightarrow \quad \alpha + \frac{1}{\alpha} = \frac{a^2 + b^2 + c^2 + d^2}{\det(M)}.$$

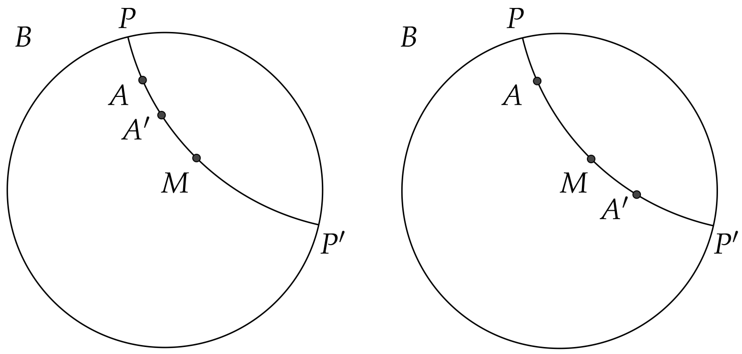

Figure 3.2: There is a different formula for the hyperbolic distance $d(A, A')$ between two points $A, A' \in B$ depending on whether they lie on the same side of the middle point M (Propositon 3.13).

Since $D(i, Mi) = \alpha \geq 1$,

$$\alpha \leq \alpha + \frac{1}{\alpha} \leq 2\alpha \quad \Rightarrow \quad \log(\alpha) \leq \log\left(\alpha + \frac{1}{\alpha}\right) \leq 0 + \log(\alpha),$$

so using (3.7) and (3.6) for the positive elements a^2, b^2, c^2 and d^2 results in the stated formula

$$\begin{aligned} d(i, Mi) &= \log(D(i, Mi)) = \log(\alpha) = \log\left(\alpha + \frac{1}{\alpha}\right) \\ &= \log\left(\frac{a^2 + b^2 + c^2 + d^2}{\det(M)}\right) \\ &= \log\left(a^2 + b^2 + c^2 + d^2\right) - \log\left(\det(M)\right) \\ &= \max\left\{\log(a^2), \log(b^2), \log(c^2), \log(d^2)\right\} - \log\left(\det(M)\right). \quad \square \end{aligned}$$

To get formulae for the hyperbolic distance $d(A, A')$ of two points A, A' in the Poincaré-disc model B of the hyperbolic plane, remember that the two points define a unique hyperbolic line that can be extended to the points P, P' 'at infinity', which lie on the Euclidean unit-circle in F^2. The line $\overline{AA'}$ is a Euclidean arc and contains a unique *midpoint* M that has the same Euclidean distance to P and P'.

Proposition 3.13 *Let $A \neq A' \in B$ form a hyperbolic line with the corresponding 'points at infinity' P, P' and with midpoint M as in figure 3.2 . Then*

$$d(A, A') = \begin{cases} \log|A'P| - \log|AP|, & \text{if } A' \text{ lies between } A \text{ and } M \\ 2\log|PP'| - \left(\log|AP| + \log|A'P'|\right), & \text{if } M \text{ lies between } A \text{ and } A', \end{cases} \tag{3.14}$$

where $|CD|$ denotes the Eudlidean distance between C and D for all $C, D \in B$.

Proof [2]
In the first case, if A' lies between A and M, by geometric considerations,

the triangle inequality and the properties (3.2) and (3.5) of the logarithm

$$\begin{aligned} & |MP'| \le |A'P'| \le |AP'| \le |PP'| \le 2|MP'| \\ \Rightarrow \quad & \log|MP'| \le \log|A'P'| \le \log|AP'| \le \log|PP'| \le \log(2) + \log|MP'| = \log|MP'| \\ \Rightarrow \quad & \log|A'P'| = \log|AP'| = \log|MP'| \end{aligned}$$

hold. Calculate

$$\begin{aligned} d(A, A') = \log(D(A, A')) &= \log\left(\frac{|A'P||AP'|}{|AP||A'P'|}\right) \\ &= \log|A'P| + \log|AP'| - \log|AP| - \log|A'P'| = \log|A'P| - \log|AP| \end{aligned}$$

using the formula for D from (1.11) and the properties of the logarithm (3.2), (3.7). In case M lies between A and A', see as before that

$$\log|A'P| = \log|MP| = \log|PP'| = \log|MP'| = \log|AP'|,$$

resulting in

$$\begin{aligned} d(A, A') &= \log|A'P| + \log|AP'| - \log|AP| - \log|A'P'| \\ &= 2\log|PP'| - \left(\log|AP| + \log|A'P'|\right). \end{aligned}$$

Note that those two possibilities are the only ones, after swapping A and A' if necessary. In the case that $A' = M$, both formulas are correct. □

Corollary 3.14 *The point $A \in B$ has hyperbolic distance 0 from the origin $O \in B$ (is in the same equivalence class as O in HF^2) if and only if*

$$\log(1 - \|A\|) = 0.$$

Proof The hyperbolic line spanned by A and O is a Euclidean line segment and $M = O$. So

$$d(A, O) = \log|OP| - \log|AP| = \log(1) - \log(1 - |AO|) = 0 - \log(1 - \|A\|)$$

implies, $d(A, O) = 0$ if and only if $\log(1 - \|A\|) = 0$. □

Proposition 3.15 *Let $A, A' \in B$ and let $s = 1 - \|A\|$ be the Euclidean distance from A to the nearest point on ∂B. Then $d(A, A') = 0$ if and only if A' lies in a Euclidean circle in B, centered at A, with radius r, where r satisfies $\log(s - r) = \log(s)$, or equivalently, $\log(1 - \frac{r}{s}) = 0$.*

Proof [2]
First note that Euclidean circles are also hyperbolic D-circles, but they may have different centers or radii. This could be proven in detail by the fact that D is invariant under rotation-subgroups of $PGL_+(2, F)$. So let the Euclidean circle around A with radius r have the hyperbolic center C. For symmetry

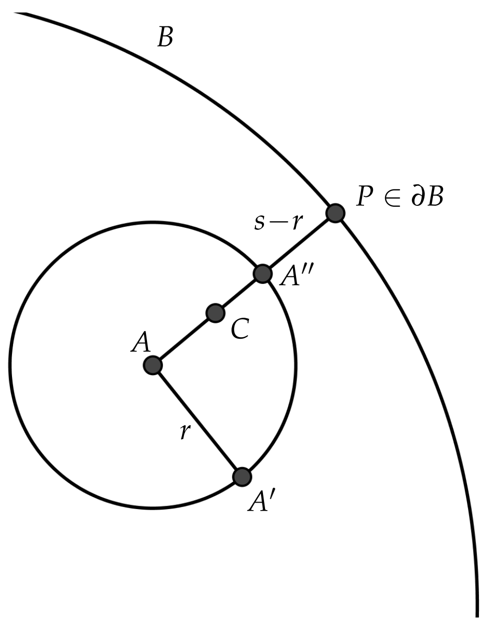

Figure 3.3: Two points $A, A' \in B$ have hyperbolic distance $d(A, A') = 0$ if and only if A' lies on the Euclidean circle around A with radius r, such that $\log(s-r) = \log(r)$. This Euclidean circle is also a hyperbolic circle around C.

reasons, C needs to be on the line $\overline{OA}$ and A lies between O and C. Let A'' be the point that lies on the circle when the line $\overline{AC}$ is prolonged and let P be its ideal end point on the boundary. For that point A'',

$$d(A, A'') = \log|PA| - \log|PA''| = \log(s) - \log(s-r)$$

holds by (3.14), so $d(A, A'') = 0$ if and only if $\log(s-r) = \log(s)$.

If $d(A, A'') = 0$, then by the triangle inequality also $0 = d(C, A'') = d(C, A')$ and $d(A, C) = 0$. So $d(A, A') \leq d(A, C) + d(C, A') = 0$.

For the other direction, assume by contradiction that $d(A, A'') \neq 0$ and $d(A, A') = 0$. Then $d(A, C) \neq 0$ or $d(C, A'') \neq 0$.

In the first case $d(A, C) \neq 0$, use the triangle inequality twice to see $0 < d(A, C) \leq d(A, A') + d(C, A') = d(C, A') \leq d(A, A') + d(A, C) = d(A, C)$, so $d(A, C) = d(C, A')$. The two points A and A' have the same hyperbolic distance from C, so they lie on the same hyperbolic circle with midpoint C, which is the Euclidean circle. But the Euclidean center A cannot lie on the circle around A.

In the second case $d(C, A'') \neq 0$, see that $d(C, A') \neq 0$ since A, A' have the same hyperbolic distance. Then a similar calculation $0 < d(C, A') \leq d(A, C) + d(A, A') = d(A, C) \leq d(A, A') + d(C, A') = d(C, A')$ shows that $d(A, C) = d(C, A') > 0$ which again cannot be, since the hyperbolic circle also has to be a Euclidean circle. □

Chapter 4

The tree of a hyperbolic plane

4.1 Λ-trees

As in the last chapter, F is a non-Archimedean, Euclidean, microbial field and TF^2 the set of equivalence classes with metric d. This metric space has the surprising property that it is a kind of tree as will be elaborated in this chapter. In graph theory, trees are connected graphs without cycles. This means that there is a unique path between two vertices. The distance between two points is the amount of edges on this path. Then the path is isometric to a closed $\mathbb{Z}$-interval. This view of paths as $\mathbb{Z}$-intervals will be used to generalize it to intervals of other subgroups of $\mathbb{R}$. The next section is concerned with this generalization, following [7] closely.

Definition 4.1 *Given a soubgroup $\Lambda \subset \mathbb{R}$, a* Λ-metric space *T is a metric space, where all distances are numbers in Λ.*

The set of equivalence classes $TF^2 = HF^2/\sim$ with metric $d = \log D$ is such a Λ-metric space for $\Lambda = \log(F^+)$. Usual trees from graph theory are $\mathbb{Z}$-metric spaces.

Definition 4.2 *Given $\lambda_1 \leq \lambda_2 \in \Lambda$, the (closed)* Λ-interval *from λ_1 to λ_2 is the set of the form $[\lambda_1, \lambda_2] = \{\lambda \in \Lambda : \lambda_1 \leq \lambda \leq \lambda_2\} \subset \Lambda$ and λ_1, λ_2 are its* endpoints*. A subspace of a Λ-metric space that is isometric to a Λ-interval is called a (closed)* Λ-segment*. A segment and its corresponding interval $[\lambda_1, \lambda_2]$ are called* nondegenerate *if $\lambda_1 < \lambda_2$. Nondegenerate segments and intervals have two* endpoints*.*

Definition 4.3 *A Λ-metric space T is a* Λ-tree *if it satisfies*

(a) For every two points in T, there is a Λ-segment in T with those endpoints.

(b) The intersection of two Λ-segments in T with one endpoint in common is again a Λ-segment.

(c) *The union of two Λ-segments, whose intersection is a single point, which is endpoint of each, is again a Λ-segment.*

Property (a) guarantees that T is 'connected' via Λ segments. The second property (b) makes sure that there are no cycles, which is the main characteristic of a tree. But it also has a more subtle implication. In fact, one source [2] assumes unique segments in (a), which already guarantees the no cycle property. It is however still necessary to have property (b) as the following example shows:

Example 4.4 *Let $\Lambda = \mathbb{Q}$. Define T to be the disjunct union of three copies of $(-\infty, \sqrt{2}) \cap \mathbb{Q} \subset \mathbb{Q}$ with a special metric d. First define the positive number*

$$d_p = \begin{cases} 4 - p^2, & \text{if } p > 0 \\ 4, & \text{if } p \leq 0 \end{cases}$$

for every $p \in (-\infty, \sqrt{2}) \cap \mathbb{Q}$. Then let

$$d(p,q) = \begin{cases} |d_p - d_q|, & \text{if } p \text{ and } q \text{ lie on the same interval} \\ d_p + d_q, & \text{if } p \text{ and } q \text{ lie on different copies of the interval} \end{cases}$$

defines a metric with values in $\mathbb{Q}$. Imagine the three intervalls to be glued together at $\sqrt{2}$. The positive definiteness and symmetry of this d are directly visible, but the triangle inequality requires some work. First, if p, q, r are all part of the same copy in that order, then $d(p,q) + d(q,r) = d(p,r)$ implies the triangle inequality. If $p \leq q$ are in one copy, but r is in another, then

$$\begin{aligned} d(p,q) &\leq d_p + d_q \leq d_p + d_q + 2d_r \\ &= d(p,r) + d(q,r) \quad \text{and} \\ d(p,r) &= d_p + d_r = d_p - d_q + d_q + d_r \\ &= d(p,q) + d(q,r) \quad \text{and} \\ d(q,r) &= d_q + d_r \leq d_p + d_r + d_p - d_q \\ &= d(q,p) + d(p,r), \quad \text{since} \quad d_q \leq d_p. \end{aligned}$$

Lastly, if all three points are part of different copies, then

$$\begin{aligned} d(p,q) &= d_p + d_q \leq d_p + d_q + d_r + d_r \\ &= d(p,r) + d(q,r). \end{aligned}$$

In this Λ-metric space T, there are segments that go from one branch to another and they are even unique for every pair of points. But if p, q, r are in different branches, the segments $\overline{pq}$ and $\overline{pr}$ have an endpoint (p) in common, but their intersection is $[p, \sqrt{2}) \cap \mathbb{Q}$, which is not a (closed) $\mathbb{Q}$-segment anymore, violating property (b). So this property (b) of Λ-trees not only guarantees the no-cycle property, but also that the branch points have to be actual points in T.

Property (c) of Λ-trees makes sure that the metric on T agrees with the path metric on the tree: If a segment $\overline{pq}$ is isometric to the interval $[\lambda_1, \lambda_2]$, then $d(p,q) = \lambda_2 - \lambda_1$.

Example 4.5 *One might be tempted to build a $\mathbb{R}$-tree by taking a subset of $\mathbb{R}^2$. For example, one might take the union of the two coordinate-axes $T = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R} \subset \mathbb{R}^2$ with the inherited Euclidean distance. This is a $\mathbb{R}$-metric space and the properties (a) and (b) are satisfied. However, the closed $\mathbb{R}$-segments $s = [0,1] \times \{0\}$ and $t = \{0\} \times [0,1]$, whose intersection is the single point $\{(0,0)\}$, do not satisfy property (c). Both intervals are isometric to the $\mathbb{R}$-interval $[0,1]$ and so the union should be isometric to $[0,2]$, with the endpoints having a distance of 2 from each other, but $d\left((0,1),(1,0)\right) = \sqrt{2} < 2$, preventing this $\mathbb{R}$-metric space from being a $\mathbb{R}$-tree.*

4.2 TF^2 as a Λ-tree

The following theorem is the main result of [2].

Theorem 4.6 *TF^2 with metric d is a Λ-tree.*

The proof of this statement requires several lemmas. Here, a simplified proof for Euclidan fields F is given. For the more general case, see [2].

Lemma 4.7 *$PGL_+(2,F)$ acts transitively on hyperbolic lines.*

Proof Let l, h be two hyperbolic lines in the Poincaré disc model B of the hyperbolic plane. To prove the statement, it is necessary to find an isometry $f \in PGL_+(2,F)$ that sends l to h. If $l = h$, then the $f = Id$ is enough. In the case that $l \neq h$, but they still meet in one point L, assume without loss of generality (proposition 1.8) that l is on the first coordinate axis and $L = 0 \in B$. Then there is another point $H \in h \setminus l$. Let $r = \|H\| \in F$ (this is possible in Euclidean fields). Then f can be the rotation that sends $r \in B$ to $H \in B$. This f will also send l to h. In the case that l does not cut h, take any two points $L \in l, H \in h$. These two points then define a new hyperbolic line g that cuts l and h in L and H. Create an f to send l to g and an f' to send g to h as above. Then $f' \circ f \in PGL_+(2,F)$ sends l to h. $\square$

Corollary 4.8 *The images of hyperbolic lines to TF^2 are d-isometric to $\Lambda = \log(F^+)$. Also, hyperbolic line segments in HF^2 project to Λ-segments in TF^2.*

Proof Use the half plane model HF^2 of the hyperbolic plane. As $PGL_+(2,F)$ acts transitively on lines, all lines are D-isometric to the line $\{iy \in HF^2 : y > 0\} \subset HF^2$, which is D-isometric to F^+. Taking logarithms results in the statement. $\square$

Lemma 4.9 *Given points $P, Q, O \in HF^2$ with $d(P,Q) = 0$. Then the images of the hyperbolic line segments $\overline{OP}$ and $\overline{OQ}$ coincide in TF^2.*

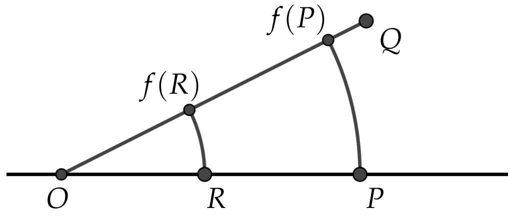

Figure 4.1: Lemma 4.9 states that if $d(P,Q) = 0$, then the line segments $\overline{OP}$ and $\overline{OQ}$ coincide in TF^2. In the proof, a rotation $f \in PGL_+(2,F)$ is introduced that sends the line OP to the line OQ.

Proof [2]
As d satisfies the triangle inequality, $d(O,P) = d(O,Q)$. If O,P,Q are hypberolically collinear, then the statement follows directly. If $d(O,P) = d(O,Q) = 0$, then all points in the hyperbolic triangle $\triangle OPQ$ have the same image in TF^2. If on the other hand $d(O,P) = d(O,Q) \neq 0$, then assume without loss of generality that O is the origin of the Poincaré disk model (as in figure 4.1). As in the proof of lemma 4.7, there is a rotation $f \in PGL_+(2,F)$ that sends the line OP to the line OQ. D and also d is $PGL_+(2,F)$ invariant, so $d(O,Q) = d(O,P) = d(O,f(P))$ and Q and $f(P)$ are on the same line (without loss of generality $f(P)$ is even on the segment OQ), so $d(Q,f(P)) = 0$. This implies that $d(P,f(P)) = 0$. For any $R \in \overline{OP}$, $D(R,f(R)) \leq D(P,f(P))$, which can be seen from formula (1.11): By going from P to R, $\|z_1 - z_2\|$ goes down, $\|1 - \overline{z_1}z_2\|$ goes up, so t becomes smaller and so does D. This inequality then implies $d(R,f(R)) = 0$. So the rotation f sends each point R from $\overline{OP}$ to a point $f(R)$ from $\overline{OQ}$ with the same image in TF^2. [2] also provides a second proof in the Klein model B_0. □

Lemma 4.10 *Let $P,Q,R \in HF^2$ with distinct images in TF^2. Then the image of the hyperbolic triangle $\triangle PQR$ in TF^2 is either three Λ-segments with one common endpoint or one Λ-segment with two points as end-points and one somewhere on the inside as shown in figure 4.2.*

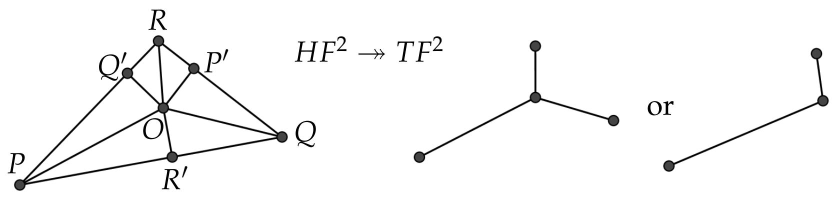

Figure 4.2: The hyperbolic triangle $\triangle PQR$ in HF^2 collapses to one of the tree like structures in TF^2. The proof of lemma 4.10 uses the fact that there is an interior point O (the incenter) and points P',Q',R' on the boundary of $\triangle PQR$ with $d(O,P') = d(O,Q') = d(O,R') = 0$. Viewing the hyperbolic plane HF^2 as a union of triangles, this can be extended to show that the quotient TF^2 is a Λ-tree.

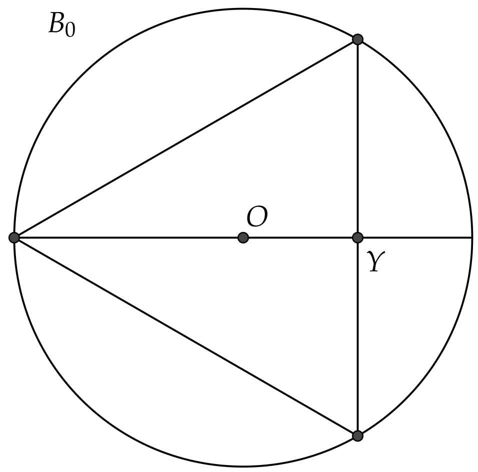

Figure 4.3: Any hyperbolic triangle in the Klein model B_0 has to be smaller than the one drawn here. In this worst case, the Euclidean distance from the closest boundary point $Y \in \partial\triangle$ is $D(O,Y) = \sqrt{(3/2)(1)(1)^{-1}(1/2)^{-1}} = \sqrt{3}$. Since $\log(\sqrt{3}) = 0$ in non-Archimedean hyperbolic planes, the Euclidean distance is $d(O,Y) = \log(D(O,Y)) = 0$.

Proof [2]
Note that there is an interior point of the triangle, for example the incenter O (being equidistant from all three sides) that can be constructed with ruler and compass and thus is in the hyperbolic plane since F is a Euclidean field. The incenter O cannot be too far away from any boundary point, namely $D(O, \partial\triangle PQR) \leq \sqrt{3}$. To see this, consider the worst case scenario of the biggest triangle in the Klein model B_0 as shown in figure 4.3. Calculating the cross ratio (1.9) results in $D(X, \partial\triangle PQR) = D(O,Y) = \sqrt{3}$.

Since $\log(\sqrt{3}) = 0$, $d(O,Y) = 0$ and because O is the incenter and has the same Euclidean distance from every side, there are points P', Q', R' on all three sides with $d(O,P') = d(O,Q') = d(O,R') = 0$. Using lemma 4.9 now implies that each of the six small triangles in figure 4.2 are sent to a closed Λ-line segment in TF^2. This results in the three closed Λ-segments with common endpoint O. If one of the points P, Q, R happens to be in the same equivalence class as O, then the triangle is sent to just one Λ-segment that somewhere contains O and the other point. $\square$

With this lemma, the tools for the proof of theorem 4.6 are assembled (following [2]).

Proof of theorem 4.6 (a) Let $[P] \neq [Q] \in TF^2$ with some representatives $P \neq Q \in HF^2$. Two points in HF^2 define a unique hyperbolic line, and the line is D-isometric to F^+ as seen in the proof of lemma 4.7. So the projection of the line to TF^2 is d-isometric to Λ. To get the Λ-interval, take the interval that corresponds to the endpoints $[P], [Q]$. $\square$

Proof of theorem 4.6 (b) Given two closed Λ-segments $\overline{[P][Q]}$ and $\overline{[P][R]}$ in TF^2, consider the triangle $\triangle PQR \subset HF^2$. As seen in lemma 4.10, the triangle either gets projected to three Λ-segments with a common endpoint $[O] \in TF^2$ in the middle, in which case the intersection of the two Λ-segments is the new Λ-segment $\overline{[P][O]}$. If the triangle is projected to a single line segment, then the intersection is just the point $[P] = [O]$ or $\overline{[P][Q]}$ is contained in $\overline{[P][R]}$ or the other way round and the intersection is the smaller of the two Λ-segments. This also implies the uniqueness of the Λ-segment. Note that $[O] \in TF^2$ is always an endpoint, thus the property that was discussed in example 4.4 is also satisfied. $\square$

Proof of theorem 4.6 (c) Let $\overline{[P][Q]}$ and $\overline{[Q][R]}$ be two Λ-segments that only intersect in $\{[Q]\}$. Then the first case in lemma 4.10 is not possible, implying that the union of the two Λ-segments indeed is again a Λ-segment $\overline{[P][R]}$.$\square$

The main step in the proof was lemma 4.10. The picture of a hyperbolic triangle in TF^2 (as in figure 4.2) shows that it is a Λ-tree. Seeing the plane as an infinite union of triangles naturally leads to all of TF^2 being a tree.

4.3 The tree of the hyperbolic plane over the Levi-Civita field

The metric space TF^2 is a Λ-tree and in this chapter, the tools for further investigations have been developed. Choosing a specific field F allows to get a better intuition about the tree. In the following, F is the Levi-Civita field over the real numbers from chapter 2.2. It is a non-Archimedean, Euclidean, microbial field with big elements such as X^{-1} and therefore its quotient TF^2 is a Λ-tree by theorem 4.6. The logarithm with basis X^{-1} always takes on rational values, so $\Lambda = \mathbb{Q}$.

Proposition 4.11 *Every point in TF^2, where F is the Levi-Civita field, has (uncountable) many* directions *(germs of closed segments that only intersect trivially at that point). In graph-theoretic language, the degree of every vertex is infinite.*

Proof Using the Poincaré disk B, the point can be taken to be $0 \in B$ without loss of generality. corollary 3.14 states that a point $A \in B$ lies in the same equivalence class as the origin if and only if $\log(1 - \|A\|) = 0$. As $\log(r) = 0$ for real numbers $r \in \mathbb{R}$, any $A \in B$ that has a real Euclidean distance from $0 \in B$ (for example $A \in (0,1) \cap \mathbb{R} \subset F[i]$) results in $\log(1 - \|A\|) = \log(1 - A) = 0$. So, a huge part of B will collapse to the same equivalence class in TF^2. Are there even any elements in a different equivalence class left? Points such as $1 - X \in B$ are extremely close to the boundary of B, and they are far enough away from $0 \in B$ to have positive hyperbolic distance, as proposition 3.13 with $A' = O$ and $P = 1 \in B$

$$\begin{aligned} d(O, A) &= \log|OP| - \log|AP| = \log(1) - \log(1 - \|A\|) \\ &= -\log(1 + (1 - X)) = -\log(X) = 1 \end{aligned}$$

shows. There is an uncountable amount of real-number-pairs (and even more in the Euclidean plane over the Levi-Civita field F) on ∂B and each of them can have their own ball of radius $2X$ around themselves. So for any real-number-pair $R \in \partial B \subset F^2$, there is $A = (1 - X)R \in B$ that has hyperbolic distance 1 from O as previously seen. As all of them are contained in the disjoint balls around the real numbered points on ∂B, all of those points A also lie in different equivalence classes, proving that there is an uncountable amount of points in TF^2 with hyperbolic distance 1 from the origin. $\square$

Any nondegenerate $\mathbb{Q}$-segment contains (countable) infinitely many points and every single one of them is a startpoint of (uncountabe) infinite other $\mathbb{Q}$-segments. This leads to the surprising property that someone standing on one point in TF^2 has more choices of where to go (uncountable infinite) than how many (countable infinite) points he can actually pass, when walking from one point to another.

4.4 Geometric properties of TF^2

One of the original motivations to consider planes over non-Archimedean fields was to investigate whether Archimedes' axiom (A) in Hilbert's system (see Appendix A) of axioms actually was required. In chapter 1 the hyperbolic plane over a general field was set up and the group of isometries $PGL_+(2, F)$ was used to be able to define congruence. Usually (in the Archimedean hyperbolic plane) this group arises from the metric on the hyperbolic plane. The analogous construction of the metric for non-Archimedean, microbial fields naturally leads to the Λ-tree TF^2. Does it satisfy Hilberts axioms of a neutral geometry? If lines are defined to be subsets of TF^2, which are isometric to Λ, then the first property (I1) already is violated: For two distinct points, there might be many lines going through them. However, every line contains at least two points (I2) and there are three non-collinear points (I3). Also the first axioms of betweenness (B1 - B3) work out. But Pasch's axiom (B4) is again not necessarily true for these lines. The axioms of congruence (C1 - C6) are satisfied.

Lines in TF^2 are instead taken to be images of hyperbolic lines in HF^2. By lemma 4.9, the uniqueness of the hyperbolic line in HF^2 transfers over to lines in TF^2. The other axioms (I2 - I3), (B1 - B4) (including Pasch's (B4)) and (C1 - C6) are also satisfied. Since the hyperbolic distances between points lie in $\Lambda \subset \mathbb{R}$, the Archimedean axiom (A) is satisfied although the base field is non-Archimedean. It turns out that the tree of a non-Archimedean Euclidean hyperbolic plane indeed satisfies the axioms of neutral geometry and surprisingly even Archimedes' axiom (A).

Chapter 5

Conclusion

We successfully constructed a hyperbolic plane over non-Archimedean base fields and investigated some of its properties. In chapter 1, we saw that it is possible to have a geometry in Hilbert's sense without defining a distance on the hyperbolic plane over a general Euclidean ordered field. Instead it was enough to consider the cross ratio, a purely algebraic notion, and the group isometries $PGL_+(2, F)$, which preserve the cross ratio. Chapter 2 provides plenty of examples and counter examples for the various properties. Most notably we constructed the non-Archimedean Euclidean microbial Levi-Civita field, which served as an example for the remaining chapters. In the search for more similarities to the Archimedean case, we introduced the logarithm of microbial fields in chapter 3, which allowed us to define the pseudo-metric d on the hyperbolic plane HF^2. Factoring out equivalence classes resulted in the Λ-metric space TF^2, which is the closest we get to a hyperbolic metric space in the Archimedean case.

In chapter 4, TF^2 was shown to be a Λ-tree. The main step was lemma 4.10, which specified how triangles look like in TF^2. The process of going from a triangle in Euclidean geometry to a hyperbolic triangle intuitively feels like squeezing the triangle (figure 5). The analogy for positively curved spaces is that triangles look like the Euclidean triangles were inflated. Interpreting lemma 4.10 under this point of view means that the triangle was even more squeezed. So to speak, the triangle in the tree of a non-Archimedean hyperbolic plane is the most hyperbolic triangle, it is not possible to squeeze it further. Although this is a pleasing point of view, it is not really backed by the math presented here and manifolds usually do not have infinitely negative curvature.

It is notable that we arrived at the results using geometric tools. The connection of non-Archimedean hyperbolic planes to Λ-trees was established by [8] using valuation rings O, where the associated Λ-tree consists of homothety classes of O-lattices. The geometrical point of view first considered by Brum-

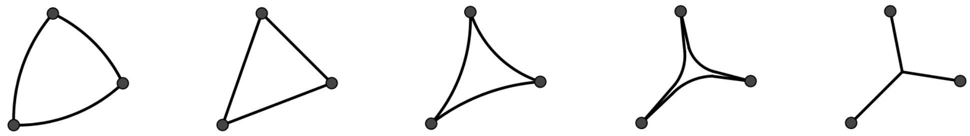

Figure 5.1: The appearance of triangles depends on the curvature of their surface. On the left is a triangle in a surface with positive curvature. Next, is the Euclidean triangle, followed by a hyperbolic triangle. Imagining this as a process of squeezing the triangle leads to the possibility to squeeze so much that there is no interior left. This corresponds to what happens to a triangle in TF^2.

fiel [2] and presented in this thesis achieves the same result without using some of the more advanced algebraic tools. In addition to Brumfiel's paper we gave various examples of ordered fields, discussed the necessity of the properties of Λ-trees and considered the connection to Hilbert's formulation of axiomatic geometry.

5.1 Acknowledgements

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Appendix A

Hilbert's axioms for geometry

Here are Hilbert's axioms for geometry and a series of definitions that arise in this context. [5] contains a wonderful discussion of these axioms.

A *neutral geometry* (or *Hilbert plane*) is a set of points $\mathfrak{P}$ with a set of lines $\mathfrak{L} \subset 2^{\mathfrak{P}}$ satisfying the following axioms (I1 - I3, B1 - B4, C1 - C6).

A *Euclidean plane* is a neutral geometry satisfying (P) and (E).

A *hyperbolic plane* is a neutral geometry satisfying (L).

Axioms of incidence

(I1) For any two distinct points A, B there exists a unique line l containing A and B.

(I2) Every line contains at least two points.

(I3) There exist three noncollinear points (that is, three points not all contained in a single line).

Axioms of betweenness
Postulate a relation between sets of three points A, B, C, called *B is between A and C* (or $A * B * C$) with the properties

(B1) If $A * B * C$, then A, B, C are three distinct points on a line, and also $C * B * A$.

(B2) For any two points $A \neq B$ there exists a point C such that $A * B * C$.

(B3) Given three distinct points on a line, one and only one of them is between the other two.

(B4) (Pasch). Let A, B, C be three non-collinear points, and let l be a line not containing any of A, B, C. If l contains a point D such that $A * D * B$, then it must also contain a point P with either $B * P * C$ or $A * P * C$.

This group of axioms (B1 - B4) allows the definition of a *line segment* $\overline{AB}$ as the set consisting of the points A and B and of all the points that lie in between A and B. A *ray* $\overrightarrow{AB}$ is the set that contains A and B, as well as all points P that satisfy $A * P * B$ or $A * B * P$. An *angle* $\sphericalangle ABC$ is the union of two rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$. The next set of axioms postulates a relation *congruence* for line segments $\overline{AB} \cong \overline{CD}$ (C1 - C3) and angles $\sphericalangle ABC \cong \sphericalangle DEF$ (C4 - C6).

Axioms of congruence

(C1) Given a line segment $\overline{AB}$, and given a ray r originating at a point C, there exists a unique point D on the ray r such that $\overline{AB} \cong \overline{CD}$.

(C2) If $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \cong \overline{EF}$, then $\overline{CD} \cong \overline{EF}$. Every line segment is congruent to itself.

(C3) (Addition). Given three points A, B, C on a line satisfying $A * B * C$, and three further points D, E, F on a line satisfying $D * E * F$, if $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$, then $\overline{AC} \cong \overline{DF}$.

(C4) Given an angle $\sphericalangle BAC$ and given a ray $\overrightarrow{DF}$, there exists a unique ray $\overrightarrow{DE}$, on a given side of the line $\overline{DF}$, such that $\sphericalangle BAC \cong \sphericalangle EDF$.

(C5) For any three angles α, β, γ, if $\alpha \cong \beta$ and $\alpha \cong \gamma$, then $\beta \cong \gamma$. Every angle is congruent to itself.

(C6) (SAS). Given triangles ABC and DEF, suppose that $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$, and $\sphericalangle BAC \cong \sphericalangle EDF$. Then the two triangles are congruent, namely, $\overline{BC} \cong \overline{EF}$, $\sphericalangle ABC \cong \sphericalangle DEF$ and $\sphericalangle ACB \cong \sphericalangle DFE$.

Other notable axioms that are sometimes used in this context:

(P) (Parallel axiom, Playfair). For each point A and each line l, there is at most one line containing A that is parallel to l (two distinct lines are parallel if they have no points in common, every line is parallel to itself).

(E) (Circle-circle intersection property). Given two circles Γ, Λ, if Λ contains at least one point inside Γ, and Λ contains at least one point outside Γ, then Γ and Λ will meet.

(A) (Archimedes). Given line segments $\overline{AB}$ and $\overline{CD}$, there is a natural number n, such that n copies of $\overline{AB}$ added together will be greater than $\overline{CD}$.

(L) (Existence of limiting parallel lines). For each line l and each point A not on l, there are two rays $\overrightarrow{Aa}$, $\overrightarrow{Aa'}$ from A, not lying on the same line, and not meeting l, such that any ray $\overrightarrow{An}$ in the interior of the angle $\sphericalangle aAa'$ meets l.

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